UNIVERSITY OF CROSS RIVER STATE

FACULTY OF ENGINEERING

DEPARTMENT OF MECHANICAL ENGINEERING

COURSE TITLE: ELECTRICAL MACHINES

UNIT-1
FUNDAMENTALS OF VIBRATION

Brief history of vibration

The discovery of musical instruments such as drums, whistles etc. made the vibration known and more interesting to the scientists and engineers. It was known since long that sound is related to vibration; but no mathematical relation was available. **Galileo** (1564-1642), an Italian mathematician, studied the oscillations of strings and simple pendulum. He developed mathematical relationship between the length of a pendulum and its frequency and discussed the term resonance. The Galileo and Hooke's developed relationship between the frequency and pitch of sound.

Sir Isaac Newton (1642-1727), an English mathematician, made a lot of scientific contribution towards dynamics by introducing the definition of Forces, Mass, Momentum and three Laws of motion.

Daniel Bernoulli (1700-1782) developed the equation of motion for vibrations of beams and studied the vibrating strings and discovered the principle of superposition of harmonics in free vibration.

L. Euler (1707-1783) worked on the bending vibrations of a rod and studied the dynamics of a vibrating ring. **J. B. J. Fourier** (1768-1830) was a French mathematician who made valuable contribution to the development of vibration theory. He has shown that any periodic function can be represented by a series of sines and cosines. This work of Fourier helps in analysing the experimentally obtained vibration plots analytically. Lord Rayleigh (1842-1919), an English physicist, hascomputed the approximately natural frequencies of vibrating bodies using an energy approach. The method derived by him is useful in developing the equations of motion and the technique is known as Rayleigh's method.

A lot of work has been done in vibration by many authors. About thirty years back, the vibration analysis of complex multi-degree of freedom systems was very difficult. But now with the help of finite element method and other advanced techniques the engineers are able to use computers to conduct numerically detailed vibration analysis of complex mechanical systems even having thousands degree of freedom.

Importance of the study of vibration

Most human activities involve vibration in one form or other. Example, we hear because our eardrums vibrate and see because light waves undergo vibration. Breathing is associated with the vibration of lungs and walking involves (periodic) oscillatory motion of legs and hands. We speak due to the oscillatory motion of larynges (tongue).

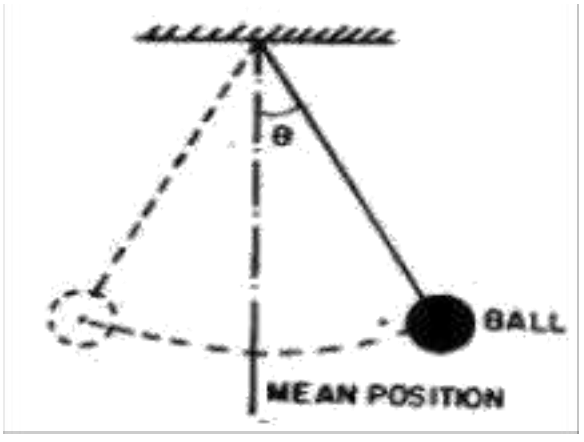
Similarly, the structures designed to support the high speed engines and turbines are subjected to vibration. Due to faulty design and poor manufacture there is unbalance in the engines which causes excessive and unpleasant stresses in the rotating system because of vibration. The vibration causes rapid wear of machine parts such as bearings and gears. Unwanted vibrations may cause loosening of parts from the machine. Because of improper design or material distribution, the wheels of locomotive can leave the track due to excessive vibration which results in accident or heavy loss. As we know that many buildings, structures and bridges fall because of vibration. If the frequency of excitation coincides with one of thenatural frequencies of the system, a condition of resonance (by the synchronous vibration of a neighbouring object)is reached, and dangerously large oscillations may occur which may result in the mechanical failure of the system. Excessive vibration is dangerous for human beings. Thus keeping in view all these devasting effects, the study of vibration is essential for a Mechanical/Aeronautical/Design engineers to minimize the vibrational effects over mechanical components by designing them suitably.

Thus, undesirable vibrations should be eliminated or reduced upto certain extent by the following methods:
1] Removing external excitation, if possible.
2] Using Shock absorbers.
3] Dynamic Absorbers.
4] Resting the system on proper vibration isolators.

Basic Concepts of Vibration

With the discovery of musical instruments like drums, the vibration became a point of interest for scientists and since then there has been much investigation in the field of vibration. All bodies having mass and elasticity are capable of vibration. The mass is inherent of the body and elasticity causes relative motion among its parts. When body particles are displaced by the application of external force, the internal forces in the form of elastic energy are present in the body. These forces try to bring the body to its original position. At equilibrium position, the whole of the elastic energy is converted into kinetic energy and body continues to move in the opposite direction because of it. The whole of the kinetic energy is again converted into elastic or strain energy due to which the body again returns to the equilibrium position. In this way, vibratory motion is repeated indefinitely and exchange of energy takes place. Thus, any motion which repeats itself after an interval of time is called vibration or oscillation.

The swinging of simple pendulum as shown in fig. 1 is an example of vibration or oscillation as the motion of ball is to and fro from its mean position repeatedly. The main reasons of vibration are as follows:
1. Unbalanced centrifugal force in the system. This is caused because of non-uniform material distribution in a rotating machine element.
2. Elastic nature of the system.
3. External excitation applied on the system.
4. Winds may cause vibrations of certain systems such as electricity lines, telephone lines, etc.



Classification of Vibrations

Vibrations can be classified in several ways. Some of the important classifications are as follows:

Free Vibration: If a system, after an initial disturbance, is left to vibrate on its own, the ensuing vibration is known as free vibration. No external force acts on the system. The oscillation of a simple pendulum is an example of free vibration.

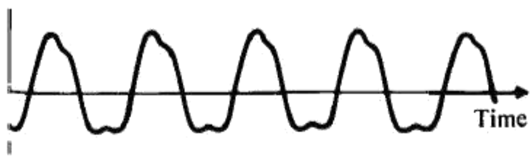
Forced Vibration: If a system is subjected to an external force, the resulting vibration is known as forced vibration. The oscillation that arises in machines such as diesel engines is an example of forced vibration.

If the frequency of the external force coincides with one of the natural frequencies of the system, this condition is known as resonance and the system undergoes dangerously large oscillations. Failures of such structures as buildings, bridges, turbines and airplane wings have been associated with the occurrence of resonance.

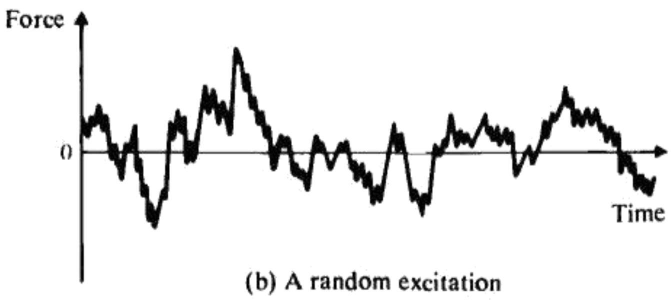
If no energy is lost or dissipated in friction or other resistance during oscillation, the vibration is known as undamped vibration. If any energy is lost in this way, on the other hand, it is called damped vibration. In many physical systems, the amount of damping is so small that it can be disregarded for most engineering purposes. However, consideration of damping becomes extremely important in analyzing vibratory systems near resonance.

If all the basic components of a vibratory system – the spring, the mass and the damper – behave linearly, the resulting vibration is known as linear vibration, on the other hand, if any of the basic components behave non linearly, the vibration is called non linear vibration. The differential equations that govern the behaviour of linear and non linear vibratory systems are linear and non linear respectively. If the vibration is linear, the principle of superposition holds, and the mathematical techniques of analysis are well developed. For non linear vibration, the superposition principle is not valid, and techniques of analysis are less well known. Since all vibratory systems tend to behave non linearly with increasing amplitude of oscillation, a knowledge of non linearly vibration is desirable in dealing with practical vibratory systems.

If the value of the excitation (force or motion) acting on a vibratory system is known at any given time, the excitation is called deterministic. The resulting vibration is known as deterministic vibration. In some cases, the excitation is non deterministic or random; the value of the excitation at a given time cannot be predicted.



(a) A deterministic (periodic) excitation



(b) A random excitation

Vibration Analysis Procedure

A vibratory system is a dynamic system for which the variables such as the excitations (inputs) and responses (output) are time-dependent. The response of vibrating system generally depends on the initial conditions as well as the external excitations. The analysis of a vibrating system usually involves mathematical modelling, derivation of the governing equations, solution of the equations, and interpretation of the results.

Step 1] Mathematical Modelling: The purpose of mathematical modelling is to represent all the important features of the system for the purpose of deriving the mathematical (or analytical) equations

governing the behaviour of the system. The mathematical model should include enough details to be able to describe the system in terms of equations without making it too complex. The mathematical model may be linear or non linear depending on the behaviour of the components of the systems. Sometimes the mathematical model is gradually improved to obtain most accurate results.

Step 2] Derivation of Governing Equations: Once the mathematical model is available, we use the principles of dynamics and derive the equations that describe the vibration of the system. The equations are usually in the form of a set of ordinary differential equations for a discrete system and partial differential equations for a continuous system. The equations may be linear or non linear depending on the behaviour of the components of the system. Several approaches are commonly used to derive the governing equations. Among them are Newton's second law of motion, d'Alembert principle and the principle of conservation of energy.

Step 3] Solution of the Governing Equations: The equations of motion must be solved to find the response of the vibrating system. Depending on the nature of the problem, we can use one of the following techniques for finding the solution: standard methods of solving differential equations, Laplace transformation methods, matrix methods, and numerical methods. If the governing equations are non linear, they can seldom be solved in closed form. Further, the solution of partial differential equations is far more involved than that of ordinary differential equations. Numerical methods, using computers, can be used to solve the equations. However, it will be difficult to draw general conclusions about the behaviour of the system using computer results.

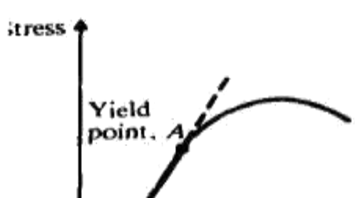
Step 4] Interpretation of the Results: The solution of the governing equations gives the displacements, velocities and accelerations of the various masses of the system. These results must be interpreted with a clear view of the purpose of the analysis and the possible design implications of the results.

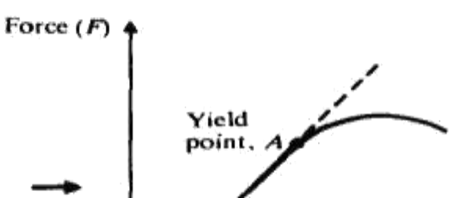
Spring Elements

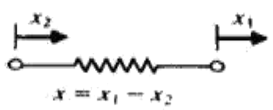
A linear spring is a type of mechanical link which is generally assumed to have negligible mass and damping. A force is developed in the spring whenever there is relative motion between the two ends of the spring. The spring force is proportional to the amount of deformation and is given by

$$F = - kx \text{ -------- } (1)$$

Where F is the *spring force*, x is the *deformation* (displacement of one end with respect to the other), and k is the *spring stiffness* or *spring constant*. The work done in deforming a spring is stored as strain or potential energy in the spring.







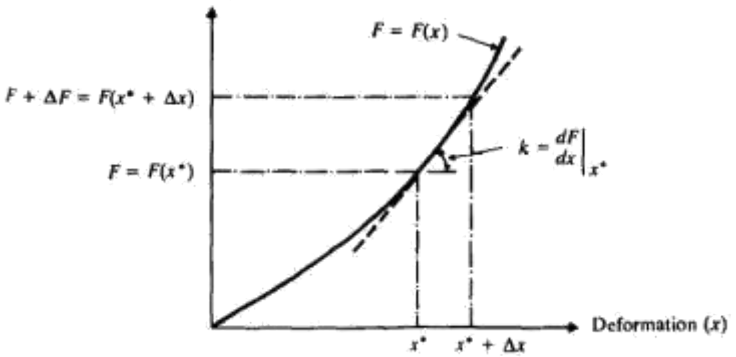
Non linearity beyond proportionality limit

Actual springs are nonlinear and follow equation 1 only up to certain deformation. Beyond a certain value of deformation (after point A in fig.), the stress exceeds the yield point of the material and the force-deformation relation becomes non linear. In many practical applications we assume the deflections to be small and make use of the linear relation in eq. 1.

Even if the force-deflection relation of a spring is non linear, as shown in fig., we often approximate it as a linear one by using a linearization process. To illustrate the linearization process, let the static equilibrium load F acting on the spring cause a deflection of *. If an instrumental force Δ is added to F, the spring deflects by an addition a quantity Δ . h + Δ can be expressed using Taylor's series expansion about the static equilibrium position * as

+Δ =F(*+Δ)=F(*)+ |(Δ) + | (Δ)² + ---------(2)

2! ² *



Linearization Process

For small values of Δ , the higher order derivative terms can be neglected to obtain

+Δ =F(*+Δ)=F(*)+ | *(x) ---------(3)

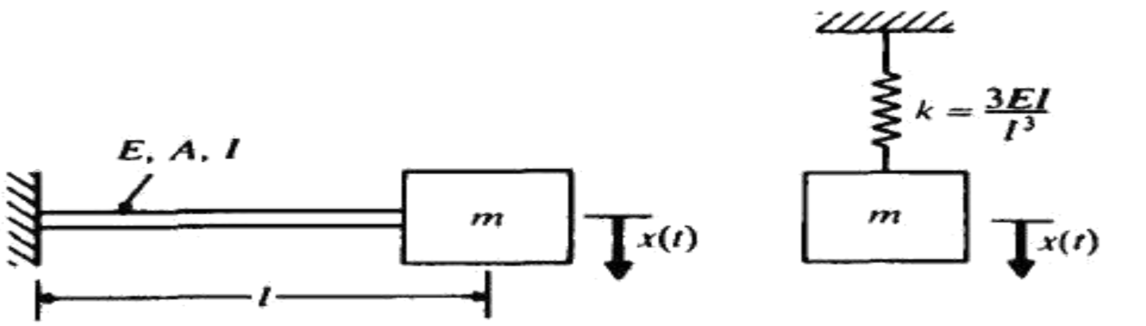
since = F(*), we can express Δ as

Δ = kΔ ---------(4)

where k is the linearized spring constant at * given by, k = | *

We may use equation 4 for simplicity, but sometimes the error involved in the approximation may be very large.

Elastic elements like beams also behave as springs. For example, consider a cantilever beam with an end mass m, as shown in fig.



(a) Actual system (b) Single degree of freedom model

We assume, for simplicity, that the mass of the beam is negligible in comparison with the mass m from strength of materials, we know that the static deflection of the beam at the free end is given by, = ---------(5)

where $W = mg$, is the weight of the mass m, E is the Young's modulus and, I is the moment of inertia of the cross section of the beam. Hence the spring constant is

k = __ = 3__3 ------- (6)

Similar results can be obtained for beams with different end conditions.

In many practical applications, several linear springs are used in combination, either in Series or in Parallel indicated below

Case (i): Springs in Series: We consider two springs connected in series, as shown in fig. since both the springs are subjected to the same force W, we have for equilibrium

W = W = 1 1 2 2 -------(7)

Where 1 and 2 are the elongations of springs 1 and 2, respectively. As the total elongation is equal to the static deflection .

1 + 2 = --------(8)

If denotes the equivalent spring constant, then for the same static deflection.

W =---------(9)

Equations 7 and 9 gives 1 1 = 2 2 =

or = and = --------(10)

Substituting these values of 1 and 2 into 8, we obtain

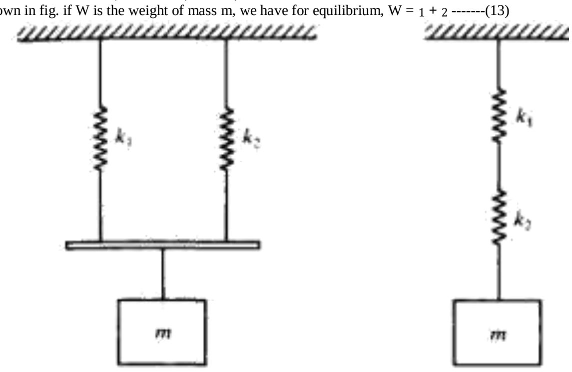
+ =

i.e. = + ------- (11)

Equations 11 can be generalized to the case of n springs in series:

= + + -------- + --------(12)

Case (ii): Springs in Parallel. Let the springs be parallel as shown in fig. if W is the weight of mass m, we have for equilibrium, W = 1 + 2 -------(13)



where is the static deflection of the mass m. if denotes the equivalent spring constant of the combination of the two springs, then for the same static deflection , we have

W = -------(14)

Equation 13 and 14 give

= 1 + 2 -------(15)

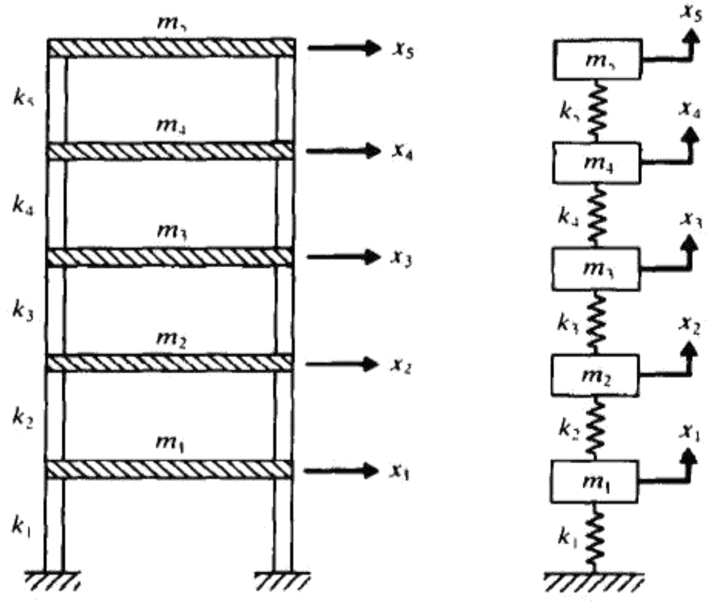
In general, if we have n springs with spring constants 1, 2, ..., in parallel, then the equivalent spring constant can be obtained:

= 1 + 2 + ------ + --------(16)

Mass or Inertia Elements

The mass or inertia element is assumed to be a rigid body; it can gain or lose kinetic energy whenever the velocity of the body changes. From Newton's second law of motion, the product of the mass and its acceleration is equal to the force applied to the mass. Work is equal to the force multiplied by the displacement in the direction of the force and the work done on a mass is stored in the form of kinetic energy of the mass.

Generally, we uses a mathematical model to represent the actual vibrating system, and there are often several possible models. The purpose of the analysis often determines which mathematical model is appropriate. Once the model is chosen, the mass or inertia elements of the system can be easily identified. For example, consider the cantilever beam with a tip mass shown in fig. For a quick and reasonably accurate analysis, the mass and damping of the beam can be disregarded; the system can be modeled as a spring-mass system as shown in fig. The tip mass m represents the mass element, and the elasticity of the beam denotes the stiffness of the spring. Next, consider a multi-story building subjected to an earthquake. Assuming that the mass of the frame is negligible compared to the masses of the floors, the building can be modeled as a multi-degree of freedom system shown in fig. The masses at the various floor levels represent the mass elements, and the elasticities of the verticalmembers denote the spring elements.



(a) (b)

Few Practical Applications:

Case (i): Translational Masses Connected by a Rigid Bar. Let the masses be attached to a rigid bar that is pivoted at one end, as shown in fig. The equivalent mass can be assumed to be located at any point along the bar. To be specific, we assume the location of the equivalent mass to be that of mass

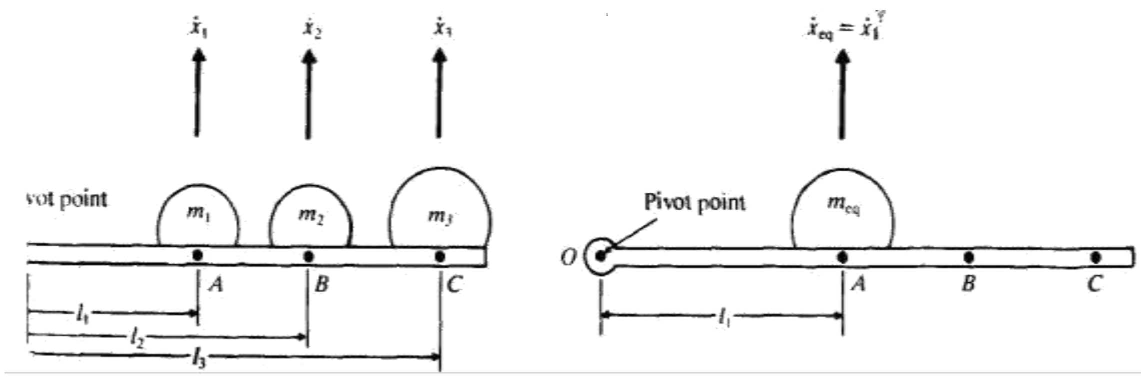
. The velocities of masses 2 ( ) and 3 ( ) can be expressed in terms of the velocity of mass

( ), by assuming small angular displacements for the bar, as

------- (17)

and

-------(18)



By equating the kinetic energy of the three mass system to that of the equivalent mass system, we obtain

1

2 1 2 1 2 1 2 -------(19)

This equation gives, in view of equation 17 and 18,

= + ( )² + ( )² -------- (20)

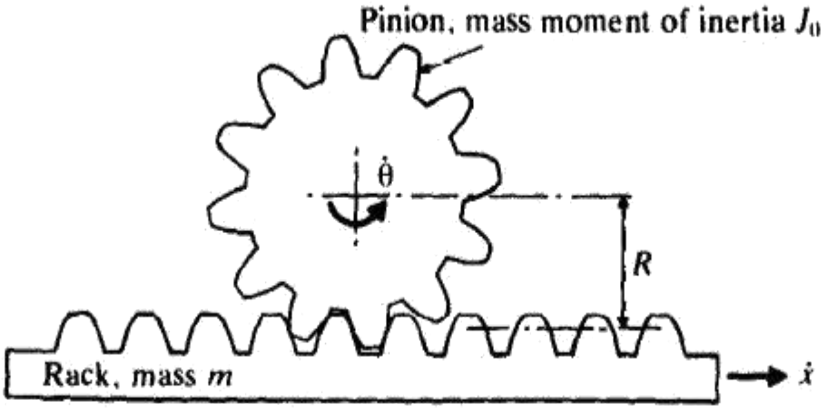
2 3

Case (ii): Translational and Rotational Masses Coupled Together. Let a mass m having a traditional velocity ˙be coupled to another mass (of mass moment of inertia 0) having a rotational velocity ˙,as in the rack and pinion arrangement shown in fig. These two masses can be combined to obtain either 1) a single equivalent translational mass or 2) a single equivalent rotational mass , as shown below.

1. Equivalent translational mass. The kinetic energy of the two masses is given by

T =

1 2 --------(22)



Translational and rotational masses in a rack and pinion arrangement

, the equivalence of T and gives

= + 0( )

that is , = + ------- (23)

2. Equivalent rotational mass. Here = *and* = , and the equivalence of T andleads to

= ( ) + 0

= +

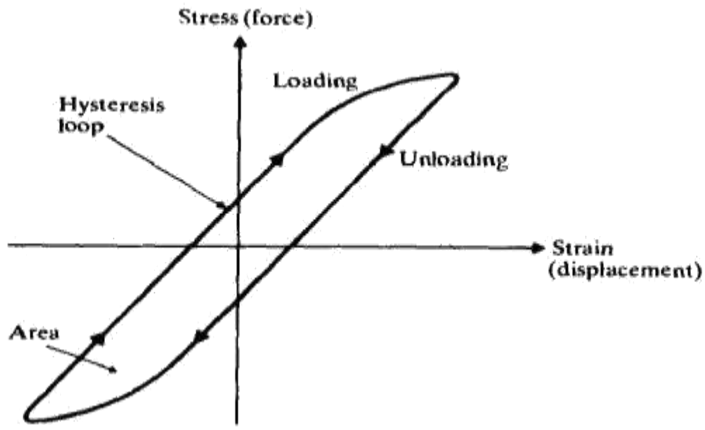
Damping Elements

In many practical systems, the vibrational energy is gradually converted to heat or sound. Due to the reduction in the energy, the response, such as the displacement of the system gradually decreases. The mechanism by which the vibrational energy is gradually converted into heat or sound is known as **Damping**. Although the amount of energy converted into heat or sound is relatively small, the consideration of damping becomes important for an accurate prediction of the vibration response of a system. A damper is assumed to have neither mass nor elasticity, and damping force exists only if there is relative velocity between the two ends of the damper. It is difficult to determine the causes of damping in practical systems. Hence damping is modeled as one or more of the following types.

Viscous Damping: It is the most commonly used damping mechanism in vibration analysis. When mechanical systems vibrate in a fluid medium such as air, gas, water, and oil, the resistance offered by the fluid to the moving body causes energy to be dissipated. In this case, the amount of dissipated energy depends on many factors, such as the size and shape of the vibrating body, the viscosity of the fluid, the frequency of vibration, and the velocity of the vibrating body. In viscous damping, the damping force is proportional to the velocity of the vibrating body. Typical examples of viscous damping include (1) fluid film between sliding surfaces. (2) fluid flow around a piston in a cylinder. (3) fluid flow through an orifice, and (4) fluid film around a journal in a bearing.

Coulomb or Dry Friction Damping: Here the damping force is constant in magnitude but opposite in direction to that of the motion of the vibrating body. It is caused due to friction between rubbing surfaces that are either dry or have insufficient lubrication.

Material or Solid or Hysteretic Damping: When materials are deformed, energy is absorbed and dissipated by the material. The effect is due to friction between the internal planes, which slip or slide as the deformations take place. When a body having material damping is subjected to vibration, the stress-strain diagram shows a hysteresis loop as shown in fig. the area of this loop dented the energy lost per cycle due to damping.



Hysteresis loop for elastic material

Definitions

Periodic Motion: A motion which repeats itself after equal intervals of time.

Time Period: Time taken to complete one cycle.

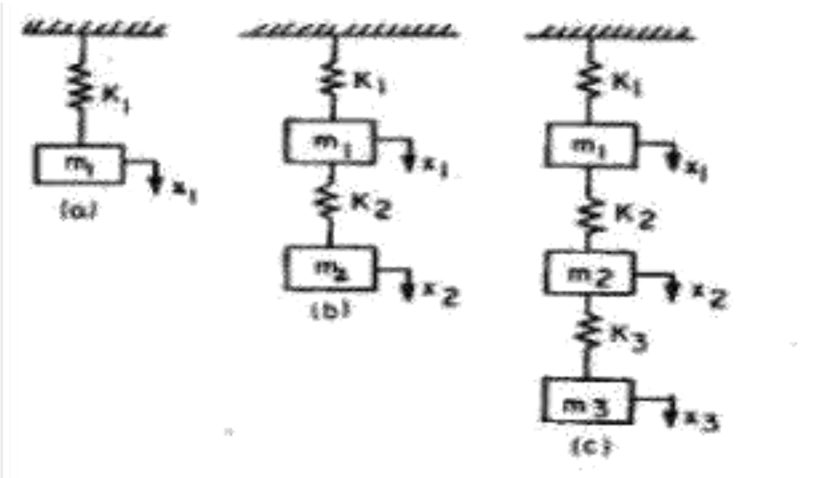
Frequency: Number of cycles per unit time.

Amplitude: The maximum displacement of a vibrating body from its equilibrium position.

Natural Frequency: when no external force acts on the system after giving it an initial displacement, the body vibrates. These vibrations are called free vibrations and their frequency as natural frequency. It is expressed in rad/sec or Hertz.

Fundamental Mode of Vibration: The fundamental mode of vibration of a system in the mode having the lowest natural frequency.

Degree of Freedom: The minimum number of independent coordinates required to specify the motion of a system at any instant is known as degrees of freedom of the system.In general, it is equal to the number of independent displacements that are possible. The one, two and three degrees of freedom systems are shown in figure.



Simple Harmonic Motion: The motion of a body to and fro about a fixed point is called simple harmonic motion. The motion is periodic, and its acceleration is always directed towards the mean position and is proportional to its distance from mean position.

Let a body having simple harmonic motion is represented by the

equation x = A sin --------(1)

˙= A -------(2)

¨= - A 2 --------(3)

or ¨= - 2 --------(4)

where x, ˙and ¨represent the displacement, velocity and acceleration of the body respectively.

Phase Difference: Suppose there are two vectors 1and 2having frequencies rad/sec each. The vibrating motions can be expressed as

1 = 1

2 = 2 sin(+ Ø) ------ (5)

Resonance: When the frequency of external excitation is equal to the natural frequency of a vibrating body, the amplitude of vibration becomes excessively large. This concept is known as resonance.

Mechanical Systems: The systems consisting of mass, stiffness and damping are known as mechanical systems.

Methods of Vibration Analysis

Some of the methods of vibration analysis are discussed here;

Energy Method:

According to this method the sum of the energies associated with the system is constant;

i.e., Kinetic Energy + Potential Energy = Constant, or (K.E. + P.E.) = Constant

1 2 1 2
(2 ˙+ 2) = 0

m˙¨+ ˙= 0

or m¨+ kx = 0 --------- (6)

This is the equation of motion.

If the motion is simple harmonic given as,

x = A sin

So, ¨= - A ² sin

Then - mA ² sin = + kA sin = 0 ---------(7)

Thus, = √ rad/sec, or f = 2 ¹ √ = Hz -------(8)

Rayleigh's Method: This method is the extension of energy method. The method is based on the principle that the total energy of a vibrating system is equal to the maximum potential energy.

At any moment total energy is either the kinetic energy or potential energy or the sum of the both. Let us say that the total energy is kinetic energy which is expressed as,

(. .) = (¹₂ ˙²) = ¹₂ ( )²

(. .) = (¹₂ ²) = ¹₂ ²

So, ( )² = ²

² = k

= √ = = 2 ¹ √ = Hz --------(9)

Equilibrium Method: According to this method the algebraic sum of the forces and moments acting on the system must be zero. If the external force acting on the system is F, spring force kx, damping force c ˙ and inertia force m ¨,then the equation of motion can be written as

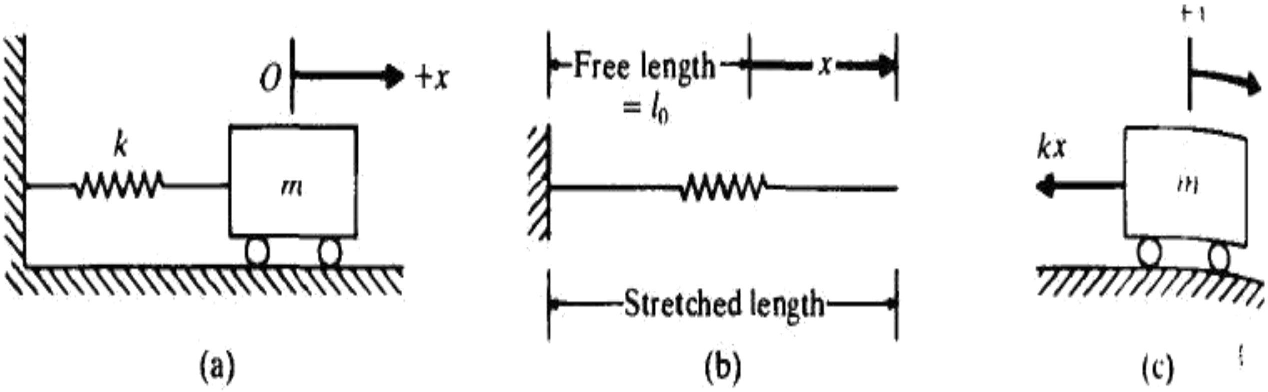
m¨+ c˙+kx = F --------(10)

FREE VIBRATION OF SINGLE DEGREE OF FREEDOM SYSTEMS

Introduction

Figure shows a spring-mass system that represents the simplest possible vibratory system. It is called a single degree of freedom system, since one coordinate (x) is sufficient to specify the position of the mass at any time. There is no external force applied to the mass; hence the motion resulting from an initial disturbance will be a free vibration. Since there is no element that causes dissipation of energy during the motion of the mass, the amplitude of motion remains constant with time; it is an **undamped system**.

In actual practice, except in a vacuum, the amplitude of free vibration diminishes gradually over time, due to the resistance offered by the surrounding medium (like air). Such vibrations are said to be **damped**.



Spring-mass System in Horizontal Position: Consider the undamped single degree of freedom system shown in figure (previous slide). The mass supported on frictionless rollers and can have translatory motion in the horizontal direction. The unstretched length of the spring is 0. Let the mass be displaced a distance +x from its rest position. This results in a spring force kx, as shown in figure. Newton's second law states that, mass X acceleration = Resultant force on the mass ------(1)

The applications of equation 1 to the mass **m** yields the equation of motion, m ̈= - kx

or, m ̈+ kx = 0 -------(2)

Where ̈= 22 is the acceleration of the mass.

Spring-mass System in Vertical Position: Consider the configuration of the spring-mass system shown in figure (in next slide). The mass hangs at the lower end of a spring, which in turn is attached to a rigid support at its upper end. At rest the mass will hang in a position called the static equilibrium position, in which the upward spring force exactly balances the downward gravitational force on the mass. In this position the length of the spring is **+** , where is the static deflection – the elongation due to the weight **W** of the mass **m**. from figure, we find that, forstatic equilibrium,

W = mg = k -------(3)

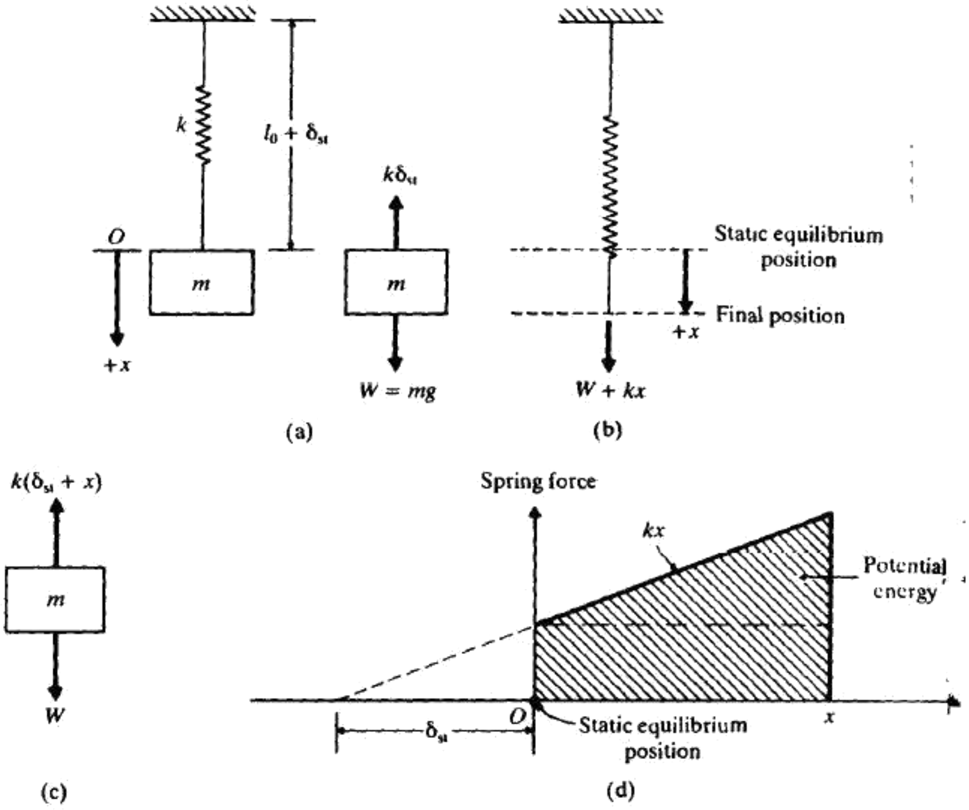
Where g, is the acceleration due to gravity. Let the mass be deflected a distance **+x**; from its static equilibrium position; thenthe spring force is **– k(x +)**, as shown in figure. The applicationof Newton's second law of motion to the mass **m** gives

m ̈= – k(x +) + W

and since k = W, we obtain

m ̈+ kx = 0 --------(4)

Equations 2 and 4 are identical. This indicates that when a mass moves in a vertical direction, we can ignore its weight, provided we measure x from its static equilibrium position.



Equation 2 can also be derived by using the conservation of energy principle. To apply this principle, first note that the system shown in figure is conservative, since there is no energy dissipation due to damping. During vibration, the energy of the system is partly kinetic and partly potential. The kinetic energy T is stored in the mass by virtue of its velocity, and the potential energy U is stored in the spring by virtue of its elastic deformation. Due to the conservation of energy, we have

T + U = constant

or (+) = 0 --------(5)

The kinetic and potential energies are given by

T = 1/2 2 --------(6)

and U = 1/2 k 2 --------- (7)

Substitutions of equation 6 and 7 into equation 5 yields the desired equation

m + kx = 0 ------(2)

The solution of equation 2 can be found by assuming

x(t) = C-------- (8)

where C and s are constants to be determined. Substitution of equation 8 into equation 2 gives

C(m 2 + k) = 0

Since C cannot be zero, we have

m 2 + k = 0 ------(9)

And hence, s = ±(− 1/2 = ± -------- (10)

where $i = (-1)^{1/2}$ and,

1/2 ----------(11)

Equation 9 is called the auxiliary or the characteristic equation corresponding to the differential equation 2. the two values of ***s*** given by equation 10 are the roots of the characteristic equation, also known as the eigen values or the characteristic values of the problem. Since both values of ***s*** satisfy equation 9, the general solution of equation 2 can be expressed as

x(t) = 1 + 2 -------(12)

where 1and 2 are constants. By using the identities

± = cos ± sin

Equation 12 can be written as x(t) = 1 cos + 2 sin --------(13)

where 1 and 2 are new constants. The constants 1 and 2 or 1and 2 can be determined from the initial conditions of the system. If the values of displacement x(t) and velocity (t) = ( ) are specified as 0 and at t = 0, we have, from equation 13,

x(t = 0) = 1 = 0

(t = 0) = 2 = ---------(14)

Hence,= 0 and 2 = . Thus, the solution of equation 2 subject to the initial conditions of equations 14 is given by

x(t) = 0 cos+ sin--------(15)

Equations 12, 13 and 15 are harmonic functions of time. The motion is symmetric about the equilibrium position of the mass **m**. The velocity is a maximum and the acceleration is zero each time the mass passes through this position. At the extreme displacements the velocity is zero and the acceleration is a maximum. Since this represents simple harmonic motion, the spring-mass system itself is called a harmonic oscillator. The quantity , given by 11, represents the natural frequency of vibration of the system.

Equation 13 can be expressed in a different form by introducing the notation

$_1 = \cos\emptyset$

Where A and Ø are the new constants which can be expressed in terms of $_1$ and $_2$ as

$_2 = \sin\emptyset$ --------(16)

$^2 \quad ^2 \quad ^{1/2} = \quad ^2 \quad ^{2\;1/2}$ = Amplitude

= Phase angle--------(17)

Introducing equation 16 into 13, the solution can be written as

x(t) = A cos (− Ø) -------- (18)

Note the following aspects of the spring-mass system:

1] If the spring-mass system is in a vertical position, the circular natural frequency can be expressed as

$^{1/2}$ --------(19)

The spring constant k can be expressed in terms of the mass ***m*** from equation 19 as

k = = --------(20)

Substitution of equation 20 into 11 yields,

$^{1/2}$ --------(21)

Hence the natural frequency in cycles per second and the natural period are given by,

$^{1/2}$ ---------(22)

$^{1/2}$ --------(23)

Thus, when the mass vibrate in a vertical direction. We can compute the natural frequency and the period of vibration by simply measuring the static deflection . It is not necessary that we know the spring stiffness k and the mass m.

2] from equation 18, the velocity ( ) and the acceleration (t) of the mass ***m*** at time ***t*** can be obtained as;

( ) = ( ) = - sin(− Ø) =cos(− Ø +)

(t) = ( ) = − ^{2} cos(− Ø) = ^{2} cos(− Ø +) --------(24)

Equation 24 shows that the velocity leads the displacement by and the acceleration leads the displacement by .

3] if the initial displacement ($_0$) is zero, equation 24 becomes

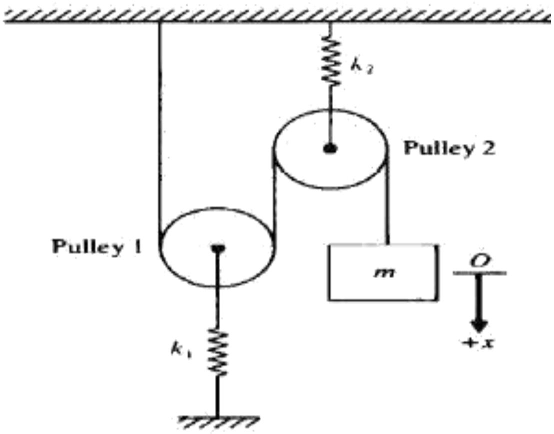
x(t) = cos(−) = ( ) --------(25)

x(t) = $_0$ --------(26)

Problem] Determine the natural frequency of the system shown in figure. Assume the pulleys to be frictionless and of negligible mass.

Solution: since the pulleys are frictionless and massless, the tension in the rope is constant and is equal to the weight W of the mass m. thus, the upward force acting on pulley 1 is 2W, and the downward force acting on pulley 2 is 2W. The center of pulley 1 moves up by a distance 2W/ $_1$ and the center of pulley 2 moves down by 2W/ $_2$. Thus, the total movements of the mass m is;

2(+)



as the rope on either side of the pulley is free to move the mass downward. If denotes the equivalent spring constant of the system;

= net displacement of the mass

$=4($ $+$ $) =$

$=$

If the equation of motion of the mass is written as

$x = 0$

The natural frequency is given by

$^{1/2} =$ $^{1/2}$ rad/sec

or, $=$ $=$ $(1 + 2)$ $^{1/2}$ cycles/sec

Free Vibration of an Undamped Torsional System

If a rigid body oscillates about a specific reference axis, the resulting motion is called. **torsional vibration.** In this case displacement of the body is measured in terms of an angular coordinate. In a torsional vibration problem, the restoring moment may be due to the torsion of an elastic member or to the unbalanced moment of a force or couple.

Figure shows a disc, which has a polar mass moment of inertia $_0$, mounted at one end of solid circular shaft, the other end of which is fixed. Let the angular rotation of the disc about the axis of the shaft be ; also represents the angle of twist of the shaft. From the theory of torsion of circular shafts, we have the relation

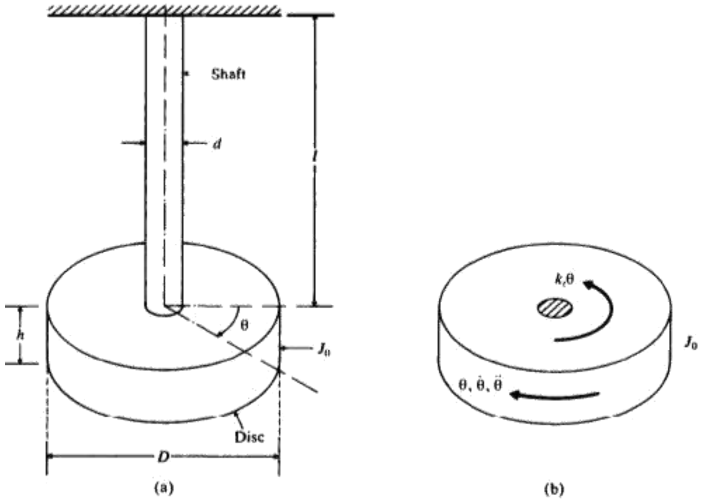
$=$ --------(27)

Where is the torque that produces the twist , G is the shear modulus, *l* is the length of the shaft, J is the polar moment of inertia of the cross section of the shaft given by

$J = \frac{\ \ ^4}{32}$ -----(28)

And d is the diameter of the shaft. If the disc is displaced by from its equilibrium position, the shaft provides a restoring torque of magnitude . Thus the shaft acts as a torsional spring with a torsional spring constant.

$=$ $=$ $=$ $\frac{\ \ ^4}{32}$ ------- (29)



The equation of angular motion of the disc about its axis can be derived by using Newton's second law or the principle of conservation of energy. By considering the free body diagram of the disc, we can derive the equation of motion by applying Newton's second law of motion:

= 0 --------(30)

Which can be seen to be identical to equation 2 if the polar mass moment of inertia $_0$ the angular displacement , and the torsional spring constant are replaced by the mass m, the displacement x, and the linear spring constant k, respectively. Thus the natural circular frequency of the torsional system is;

$= ()^{1/2}$ ----(31)

And the period and natural frequency of vibration in cycles per second are

$= 2\ ()^{1/2}$ ----(32)

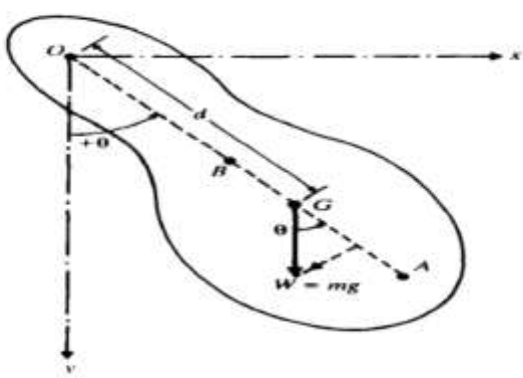
$= \frac{1}{}\ ()^{1/2}$ --------(33)

Problem] Any rigid body pivoted at a point other than its center of mass will oscillate about the pin point under its own gravitational force. Such a system is known as a compound pendulum (as shown in figure). Find the natural frequency of such a system.

Solution: Let O be the point of superposition and G be the center of mass of the compound pendulum, as shown in figure. Let the rigid body oscillate in the xy plane so that coordinate can be used to describe its motion. Let d denote the distance between O and $_0$ the mass moment of inertia of the body about the z-axis (perpendicular to both x and y). For a displacement , the restoring torque (due to the weight of the body W) (W d sin) and the equation of motion is

--------(34)

For a small angles of oscillation, sin , hence equation can be expressed as



This gives the natural frequency of the compound pendulum:

$$= (\quad — \quad)^{1/2} = (\quad — \quad)^{1/2} \text{ --------(36)}$$

Comparing equation 36 with the natural frequency of a simple pendulum, $= (/)^{1/2}$, we can find the length of the equivalent simple pendulum:

$$= \quad — \text{ --------- (37)}$$

If $_0$ is replaced by $_0^2$ becomes where $_0$ is the radius of gyration of the body about O, equation 36 and 37

$$= (\quad —)^{1/2} \text{ --------(38)}$$

$$= (_0) \text{ --------- (39)}$$

If denotes the radius of gyration of the body about G, we have

$$_0^2 = {}^2 + {}^2 \text{ -------- (40)}$$

and equation 39 becomes,

$$= (\quad — +) \text{--------(41)}$$

If the line OG is extended to point A such that =

$$— ^2 \text{ ------- (42)}$$

Equation 41 becomes,

$$= + = \text{ --------- (43)}$$

Hence, from equation 35,is given by,

$$— ^{1/2} \quad — \quad = (— ^{1/2} \text{ ------- (44)}$$

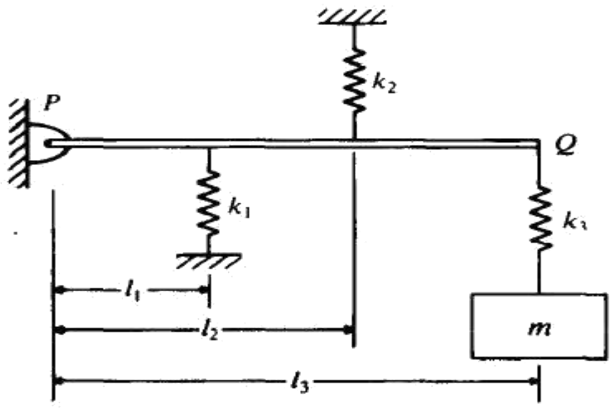
This equations shows that, no matter whether the body is pivoted from O and A, its natural frequency is the same. The point A is called the ***center of percussion***.

Problem Based on Unit 1

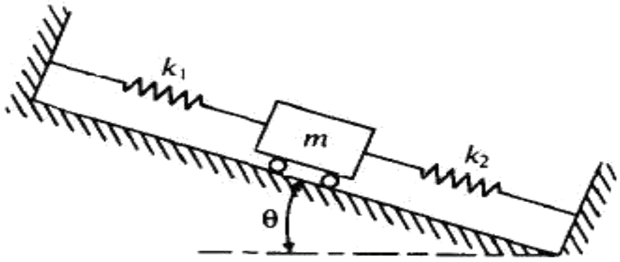
1] A spring-mass system has a natural period of 0.21 sec. what will be the new period if the spring constant is (i) increased by 50% and (ii) decreased by 50%?

2] A spring-mass system has a natural frequency of 10 Hz. When the spring constant is reduced by 800 N/m, the frequency is altered by 45%. Find the mass and spring constant of the original system.

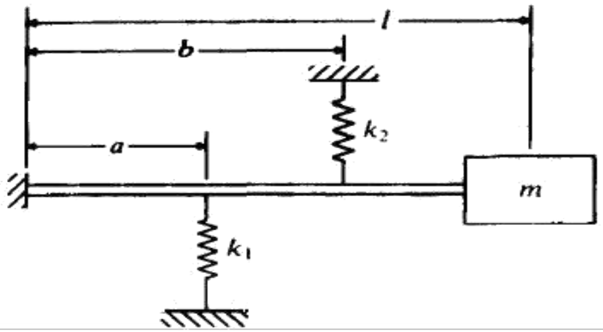
3] Three springs and a mass are attached to a rigid, weightless, bar PQ as shown in figure. Find the natural frequency of vibration of the system.



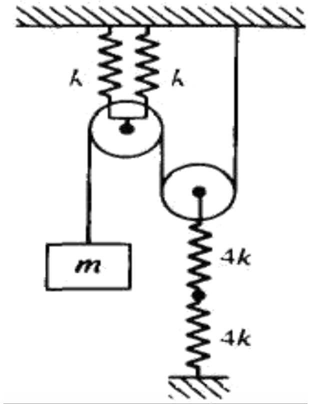
4] Find the natural frequency of vibration of a spring-mass system arranged on an inclined plane, as shown in figure.



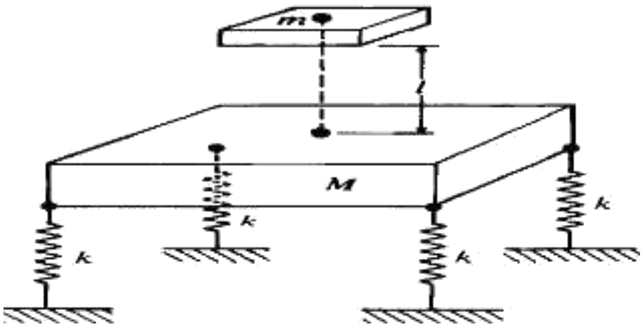
5] Find the natural frequency of the system shown in figure with and without the springs 1 and 2 in the middle of the elastic beam.



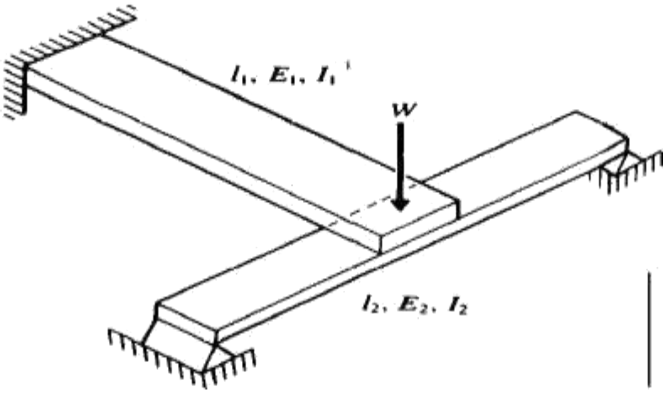
6] Find the natural frequency of the pulley system shown in figure by neglecting the friction and the masses of the pulleys.



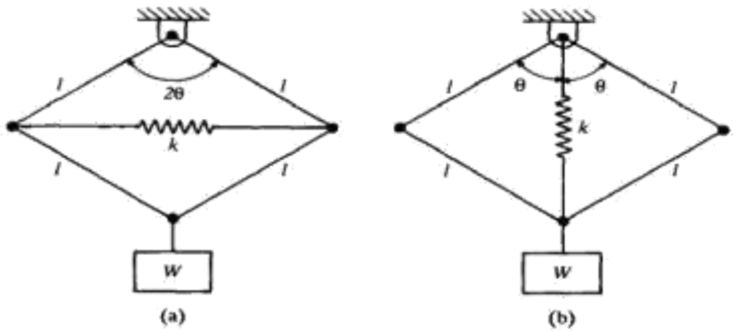
7] A rigid block of mass M is mounted on four elastic supports as shown in figure. A mass m drops from a height l and adheres to the rigid block without rebounding. If the spring constant of each elastic support is k, find the natural frequency of vibration of the system (a) without the mass m, and (b) with the mass m. Also, find the resulting motion of the system in case (b).



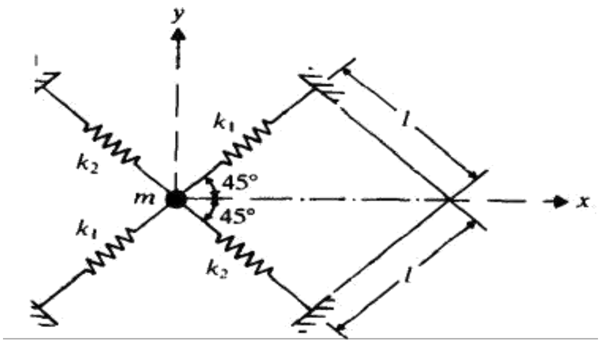
8] Derive the expression for the natural frequency of the system shown in figure. Note that the load W is applied at the tip of beam 1 and midpoint of beam 2.



9] The natural frequency of a spring-mass system is found to be 2 Hz. When an additional mass of 1 kg is added to the original mass m, the natural frequency is reduced to 1 Hz. Find the spring constant k and the mass m.



10] Four weightless rigid links and a spring are arranged to support a weight W in two different ways as shown in figure. Determine the natural frequencies of vibration of the two arrangement.



11] Figure shows a small mass 'm' restrained by four linearly elastic springs, each of which has an unstretched length *l,* and an angle of orientation of 45˚ with respect to the x-axis. Determine the equation of motion for small displacements of the mass in the x-direction.

UNIT-2

HARMONICALLY EXITED VIBRATIONS

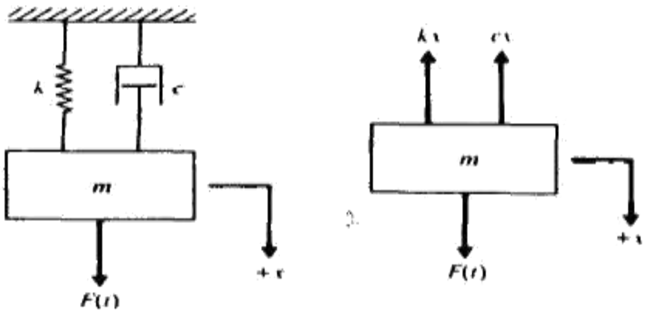
Introduction

A dynamic system is often subjected to some type of external force or excitation, called the *forcing* or *exciting function*. This excitation is usually time-dependent. It may be *harmonic*, *non-harmonic* but *periodic*, *non-periodic*, or *random* in nature. The response of a system to a harmonic excitation is called *harmonic response*. The non-periodic excitation may have a long or short duration. The response of a dynamic system to suddenly applied non-periodic excitations is called *transient response*.

Let us suppose the dynamic response of a single degree of freedom system under harmonic excitations of the form F(t) = $_0$ $^{(\ + \emptyset)}$ or F(t) = $_0$ cos(+ ∅) or F(t) = $_0$ Sin(+ ∅), where $_0$ is the amplitude, is the frequency, and ∅ is the phase angle of the harmonic excitation. The value of ∅ depends on the value of F(t) at t = 0 and is usually taken to be zero. Under a harmonic excitation, the response of the system will also be harmonic. If the frequency of excitation coincides with the natural frequency of the system, the response of the system will be very large. This condition, known as *resonance*, is to be avoided to prevent failure of the system.

Equation of Motion

If a force F(t) acts on a viscosity damped spring-mass system as shown in figure. The equation of motion can be obtained using Newton's second Law:



(a) (b) Free body diagram

m ̈+ ̇+ = ( ) -------(1)

Since this equation is homogeneous, its general solution x(t) is given by the sum of the homogeneous solution, $_h$( ), and the particular solution, ( ).
The homogeneous solution, which is the solution of the homogeneous equation; m ̈+ ̇+ = 0 ---------(2)

represents the free vibration of the system. This free vibration dies out with time under each of the three possible conditions of damping (underdamping, critical damping and overdamping), and under all possible initial conditions. Thus, the general solution of equation 1 eventually reduces to the particular solution ( ), which represents the steady-state vibration. The steady-state motion is present as long as the forcing function is present. The vibrations of homogeneous, particular, and general solutions with time for a typical case are shown in figure 2. it can be seen that $_h$( ) dies out and x(t) becomes ( ) after some time (in figure 2). The part of the motion that dies out due to damping (the free vibration part) is called transient. The rate at which the transient motion decays depends on the values of the system parameters k, c, and m.

Response of an Undamped System under Harmonic Force

First we consider an undamped system subjected to a harmonic force, for the sake of simplicity. If a force F(t) = 0 acts on the mass *m* of an undamped system, the equation of motion is,

------- (3)

The homogeneous solution of this equation is given by,

------- (4)

where is the natural frequency of the system. Because the exciting force F(t) is harmonic, the particular solution ( ) is also harmonic and has the same frequency . Thus, we assume a solution in the form

( ) = X -------(5)

where X is a constant that denotes the maximum amplitude of ( ). By substituting equation 5 into 3 and solving for X, we obtain,

Case 1. when 0 < / < 1, the denominator in equation 10 is positive and the response is given by equation 5 without change. The harmonic response of the system ( )is said to be in phase with the external force as shown in figure.

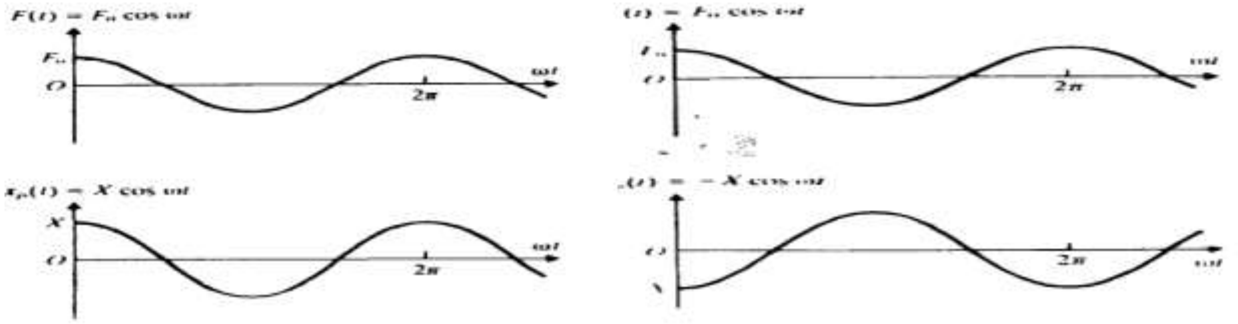
Case 2. when / > 1, the denominator in equation 10 is negative, and the steady-state solution can be expressed as ( ) = - Xcos -------- (11)

where the amplitude of motion X is redefined to be a positive quantity as

X = —— ------ (12)

The variations of F(t) and ( )with time are shown in figure. Since ( )and F(t) have opposite signs, the response is said to be 180˚ out of phase with the external force. Further, as / → , X → 0. thus, the response of the system to a harmonic force of very high frequency is close to zero.

Case 3. when / → 1, the amplitude X given by equation 10 or 12 becomes infinite. The condition, for which the forcing frequency is equal to the natural frequency of the system , is called resonance. To find the response for this



condition. We rewrite equation as

x(t) = +

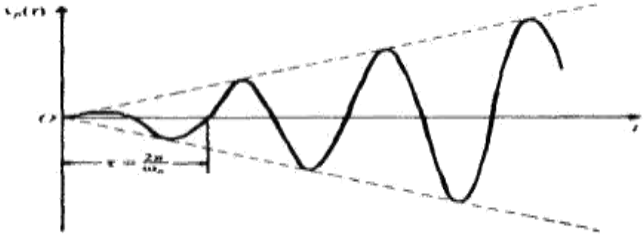
Since the last term of the equation takes an indefinite form for = , we apply L-Hospital's rule to evaluate the limit of this term;

-------- (14)

Thus the response of the system at resonance becomes;

It can be seen from equation 15 that at resonance, x(t) increases indefinitely.

The last term of equation 15 is shown in figure from which the amplitude of the response can be seen to increase linearly with time.



Response of a damped system under harmonic force

If the forcing function is given by F(t) = 0 , the equation of motion becomes

m + + = 0 ------- (16)

The particular solution of equation 16 is also expected to be harmonic; we assume it in the form

( ) = (− ∅) ------ (17)

Where X and ∅ are constants to be determined. X and ∅ denote the amplitude and phase angle of the response, respectively.

X[(k - m ^{2})cos(− ∅) - c sin(− ∅)] = 0 -------- (18)

Using the trigonometric relations;

cos(− ∅) = ∅ + ∅
sin(− ∅) = ∅ − ∅

In equation 18 and equating the coefficients of and on both sides of the resulting equation, we obtain

X[(k - m ^{2})cos∅ + c sin∅] =

X[(k - m ^{2})sin∅ - c cos∅] = 0 -------- (19)

Solution of equation 19 gives, X = -------- (20)

And ∅ = tan⁻¹( ) --------- (21)

By inserting the expressions of X and ∅ from equations 20 and 21 into equation 17. We obtain the particular solution of equation 16. figure shows typical plots of the forcing function and (steady-state) response. Dividing both the numerator and denominator of equation 20 by k and making the following substitutions,

= √ = undamped natural frequency.

= = ; = 2 ,

= = ℎ ;

= =

We obtain,

= 1 = 1 -------- (22)

And ∅ = tan⁻¹(2) = tan⁻¹(2) --------- (23)
1 − ( )^2 1 − 2

As stated, the quantity X/ is called the magnification factor, amplification factor, or amplitude ratio.

The variations of X/ and ∅ with the frequency ratio r and the damping ratio ζ are shown in figure.

The following observations can be made from (22) and (23).

1] For an undamped system (ζ = 0), equation 23 shows that the phase angle ∅ = 0 (for r < 1) or 180˚ (for r > 1) and equation 22 reduces.

2] The damping reduces the amplitude ratio for all values of the forcing frequency.

3] The reduction of the amplitude ratio in the presence of damping is very significant at or near resonance.

4] with damping, the maximum amplitude ratio occurs when = √1 − 2 ² or = √1 − 2 ² ---- ---- (24)

Which is lower than the undamped natural frequencyand the damped natural frequency= √1 − ² .

5] The maximum value of X (when r = √1 − 2 ²) is given by

( ) = 2 √1 − ² -------- (25)

And the value of X at = by ( ) = ² -------- (26)

Equation 25 can be used for the experimental determination of the measure of damping present in the system. In a vibration test, if the maximum amplitude of the response ( ) is measured, the damping ratio of the system can be found usingequation 26. conversely, if the amount of damping is known, one can make an estimate of the maximum amplitude of vibration.

6] For ζ> 1/√2. The graph of X has no peaks and for ζ = 0, there is a discontinuity at r = 1.

7] The phase angle depends on the system parameters m, c, and k and the forcing frequency but not on the amplitude 0 of the forcing function.

8] The phase angle ∅ by which the response x(t) or X lags the forcing function F(t) or 0 will be very small for small values of r. For very large values of r, phase angle approaches 180˚ asymptotically. Thus, the amplitude of vibration will be in phase with the exciting force for r << 1 and out of phase for r >> 1. The phase angle at resonance will be 90˚ for all values of damping (ζ).

9] Below resonance (<), the phase angle increases with increase in damping. Above resonance (>), the phase angle decreases with increase in damping.

Forced Vibration with Coulomb Damping

For a single degree of freedom system with Coulomb or dry friction damping, subjected to a harmonic force F(t) = 0 , the equation of motion is given by

¨+ ± = ( ) = 0 -------- (27)

Where the sign of the friction force ( ) is positive (negative) when the mass moves from left to right (right to left). The exact solution of equation 27 is quite involved. However, we can expect that if the dry friction damping force is large, the motion of the mass will be discontinuous. On the other hand, if the dry friction force is small compared to the amplitude of the applied force 0, the steady state solution is expected to be nearly harmonic. In this case, we can find an approximate solution of equation 27 by finding an equivalent viscous damping ratio. To find an equivalent viscous damping ratio, we equate the energy dissipated due to dry friction to the energy dissipated by an equivalent viscous damper during a full cycle of motion. If the amplitude of motion is denoted as X, the energy dissipated by the friction force in a quarter cycle is X. hence, in a full cycle, the energy dissipated by dry friction damping is given by

Δ = 4 ---------(28)

If the equivalent viscous damping constant is denoted as , the energy dissipated during a full cycle will be

$\Delta \; = 2$ --------(29)

By equating equations 28 and 29, we obtain

$=$ $\frac{4}{}$ -------(30)

Thus, the steady-state response is given by

$( ) = \quad (\; - \emptyset)$ --------(31)

Where the amplitude X can be found from equation:

X = ------- (32)

With

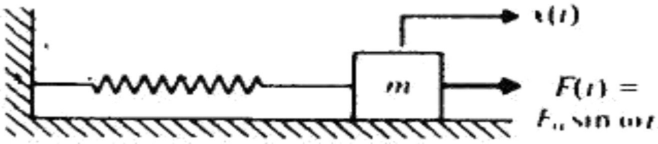
$= \quad = \quad = \quad \frac{4}{} \quad = \quad \frac{2}{}$ --------(33)

Substitution of equation 33 into equation 32 gives,

X = --------(34)

The solution of this equation gives the amplitude X as,

X = $]^{1/2}$ --------(35)



As stated earlier, equation 35 can be used only if the friction force is small compared to 0. In fact, the limiting value of the friction force can be found from equation 35. to avoid imaginary values of X, we need to have

If this condition is not satisfied, the exact analysis, is to be used. The phase angle $\emptyset$ appearing in

equation 31 can be found using, $\emptyset = \tan^{-1}$ ( ) = $\tan^{-1}$[ ] = $\tan^{-1}$[ ] -------- (36)

substituting equation 35 into equation 36 for X, we obtain

$\emptyset = \tan^{-1}$( ) --------(37)

Equation 36 shows that $\emptyset$ is a constant for a given value of / . $\emptyset$ is discontinuous at > 1.

thus equation 37 can also be expressed as

$\emptyset = \tan^{-1}$( ) -------- (38)

Equation shows that the friction serves to limit the amplitude of forced vibration for $\neq 1$. however, at resonance (= 1), the amplitude becomes infinite. This can be explained as follows. The energy

directed into the system over one cycle which it is excited harmonically at resonance is

Since equation 36 gives $\emptyset = 90^\circ$ at resonance, equation 39 becomes

$\Delta ' = 0 \quad \int_0^{2\pi/} \quad 2 \quad = \quad 0$ --------(40)

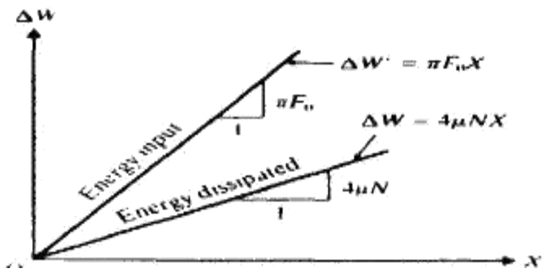
The energy dissipated from the system is given by equation. Since $_0 > 4$ for X to be real-valued. $\Delta ' > \Delta$ at resonance (see figure). Thus, more energy isdirected into the system per cycle than is dissipated per cycle. This extra energy is used to build up the amplitude of vibration. For the

non resonant condition (≠ 1) the energy input can be found from equation (39).

$$\Delta ' = \quad _0 \quad \int_0^{2} / \quad (\quad - \emptyset) \quad = \quad _0 \quad \emptyset \text{ -------- (41)}$$

Due to the presence of sin∅ in equation 41. the input energy curve in figure is made to coincide with the dissipated energy curve. So the amplitude is limited. Thus, the phase of the motion ∅ can be seen to limit the amplitude of the motion.

The periodic response of a spring-mass system with Coulomb damping subjected to base excitation.

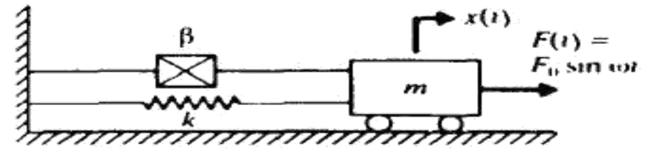


Forced Vibration with Hysteresis Damping

Consider a single degree of freedom system with hysteresis damping and to a harmonic force F(t) = $_0$ as indicated in figure. The equation of motion of the mass can be derived by using

$$m\ddot{} + \quad \dot{} + \quad = \quad _0 \qquad \text{------(1)}$$

where $\dot{} = (^h)\, \dot{}$ denotes the damping force. Although the solution of equation is quite involved for a general forcing function F(t), our interest is to find the response under a harmonic force.



Now, The steady-state solution of equation be,

$$( ) = X \sin(\quad - \quad \emptyset) \text{ ------(2)}$$

But substituting equation (2) into equation (1), we get

$$X = \frac{}{[(1- \quad)^2 \quad]^{1/2}} \text{ ------(3)}$$

$$\text{and, } \emptyset = \tan^{-1}\left[\frac{}{(1- \quad)}\right] \text{ ------(4)}$$

with these equation, we have following points:

1] The amplitude ratio, X/($_0$/k) attains its maximum value of $_0$/k at the resonant frequency (=).

2] The phase angle ∅ has a value of $\tan^{-1}$ at = 0, in the case of hysteresis damping.

3] Also, equation of motion (suppose harmonic excitation is F = $_0$)

$$m\ddot{} + \quad \dot{} + \quad = \qquad \text{-------(5)}$$

In this case, the response x(t) is also a harmonic function involving the factor . Hence, $\dot{}( )$ is given by i ( ) then equation becomes;

$$\ddot{} + \frac{}{} + = 0$$

where, the quantity k(1 + i) is called the complex stiffness or complex damping. The steady-state solution is given by the real part of,

$$\frac{}{} \quad \text{-------(7)}$$

* **Hysteresis damping model with a constant stiffness and loss factor is only applicable for harmonic excitation.**

Problem Based on unit 2

Problem 1]: A spring-mass system consists of a mass weighing 100N and a spring with a stiffness of 2000N/m. The mass is subjected to resonance by a harmonic force F(t) = 25cos N. Find the amplitude of the forced motion at the end of

(i) 1/4 cycle, (ii) 2(1/2)cycle, (iii) 5(3/4) cycles.

Problem 2]: A spring-mass system with m = 10 kg and k = 5000 N/m is subjected to a harmonic force of amplitude 250 N and frequency . If the maximum amplitude of the mass is observed to be 100 mm, find the value of .

Problem 3]: A mass 'm' is suspended from a spring of stiffness 4000 N/m and is subjected to a harmonic force having an amplitude of 100 N and a frequency of 5 Hz. The amplitude of the forced motion of the mass is observed to be 20 mm. find the value of 'm'.

Problem 4]: A weight of 50N is suspended from a spring of stiffness of 4000 N/m and is subjected to a harmonic force of amplitude 60 N and frequency 6 Hz. Find

(i) the extension of the spring due to the suspended weight.

(ii) the static displacement of the spring due to the maximum applied force and

(iii) the amplitude of forced motion of the weight.

Solution:

(i) $= \frac{}{4000} = 0.0125$

(ii) $= \frac{}{4000} = 0.015$

(iii) $= \sqrt{} = \sqrt{\frac{4000 \cdot 9.81}{50}} = 28.0143$ /

$= 2 \quad = 2 \quad 6 = 37.6992$ /

$= \left[\frac{1}{1-(\frac{}{})^2}\right] = 0.0185$

Problem 5]: A spring-mass system is subjected to a harmonic force whose frequency is close to the natural frequency of the system. If the forcing frequency is 39.8Hz, determine the period of beating?

Answer: $() = \frac{2}{2(-)} = \frac{10}{2} = 5$

UNIT-3

VIBRATION UNDER GENERAL FORCING CONDITIONS

Introduction

The vibration of a viscously damped single degree of freedom system under general forcing conditions. If the excitation is periodic but not harmonic, it can be replaced by a sum of harmonic function using the harmonic analysis procedure.

* If the system is subjected to a suddenly applied non-periodic force, the response will be transient, since steady-state vibrations are not usually produced.

Response Under a General Periodic Force

When the external force F(t) is periodic with period $= 2\ /$, it can be expanded in a Fourier series,

F(t) = — + Σ + Σ ------(1)

where, = — ∫ ( ) j = 0,1,2……..------(2)

and = — ∫ ( ) , j = 1,2……------(3)

The equation of motion of the system can be expressed as,

m¨+ ˙+ = ( ) = — + Σ + Σ ------(4)

The right hand side of this equation is a constant plus a sum of harmonic functions. Using the principle of superposition, the steady-state solution of (4) is the sum of the steady-state solution of the following equations:

m¨+ ˙+ = — ------(5)

m¨+ ˙+ = Σ -------(6)

m¨+ ˙+ = Σ -------(7)

The solution of equation 5 is given by;

( ) = — ------(8)

Also, express the solutions of equation 6 and 7 respectively

( ) = — cos(− ∅) -------- (9)

( ) = — sin(− ∅) -------- (10)

where, ∅ = tan⁻¹(—)------- (11)

and r = — ------ (12)

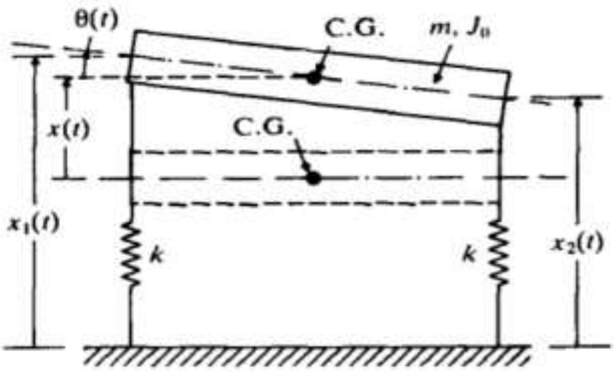
Thus, the complete steady-state solution of 4 is given by,

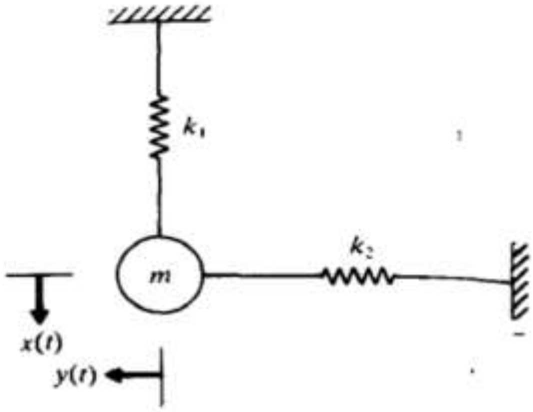
( ) = — + Σ — cos(− ∅) + Σ — sin(− ∅) ------(13)

Two Degree of Freedom Systems:

Systems that require two independent coordinates to describe their motion are called Two-degree of Freedom systems.

Example:





The general rule for the computation of the number of degrees of freedom can be stated as follows:

Number of degrees of freedom of the system = number of masses in the system X Number of possible types of motion of each mass.

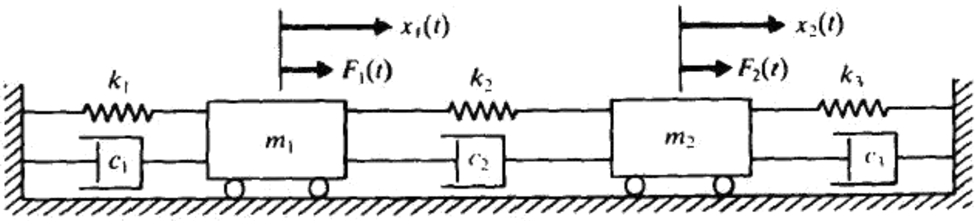
There are two equation of motion for a two degree of freedom system, one for each mass (more precisely, for each degree of freedom).

They are generally in the form of coupled differential equations – i.e., each equation involves all the coordinates. If a harmonic solution is assumed for each coordinate, the equation of motion lead to a frequency equation that gives two natural frequencies for the system. If we give suitable initial excitation, the system vibrates at one of these natural frequencies.

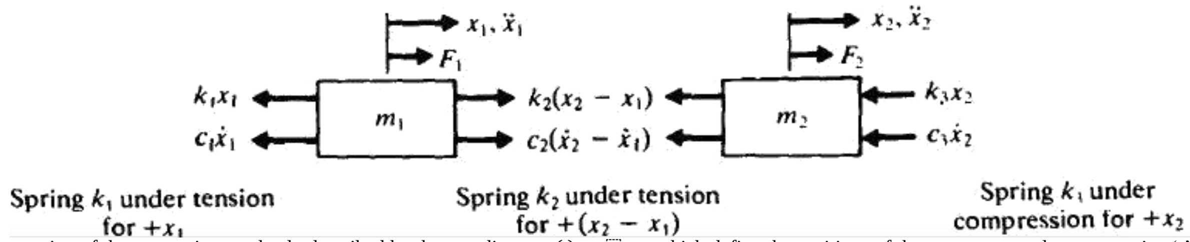
During free vibration at one of the natural frequencies, the amplitudes of the two degrees of freedom (coordinates) are related in a specific manner and the configuration is called a Normal mode, principal mode, or natural mode of vibration. Thus, a two degree of freedom system has two normal modes of vibration corresponding to the two natural frequencies.

Equations of Motion for forced Vibration:

Consider a viscously damped two degree of freedom spring-mass system as shown as figure:



(a)



Now, the motion of the system is completely described by the coordinates 1( ) 2 , which define the positions of the masses m_1 and m_2 at any time't' from the respective equilibrium positions. The external forces $F_1(t)$ and $F_2(t)$ act on the masses m_1 and m_2 are shown in figure.

By Newton's II[nd] law of motion to each of the masses gives the equation of motion:

------- (1)

------- (2)

equation (1), contains terms involving

2 [, − − 2 2], and equation (2), contains terms involving

Hence, they represent a system of two coupled differential equations. We can therefore expect that the motions of the mass m_1 will influence the motion of the mass m_2 and vice-versa.

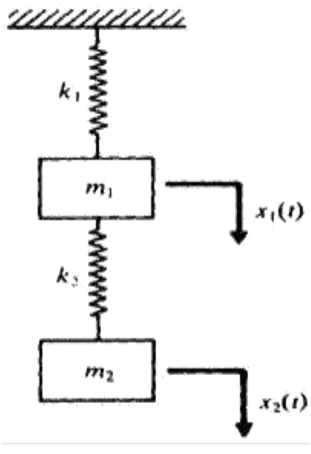
Equation (1) and (2) can be written in matrix form as;

------(3)

Where, [m], [c] and [k] are called the mass, damping and stiffness matrices respectively and are given by

and,

Problem 1]: Find the natural frequencies of the system as shown in figure with m_1 = m, m_2 = 2m, k_1 = k, and k_2 = 2k. Determine the response of the system when k = 1000 N/m, m 20 kg, and the initial values of the displacements of the masses m_1 and m_2 are 1 and -1 respectively.



Solution: equation of motion,

(1)

(2)

equation (1), give the frequency equation,

= 0 ------- (3)

roots of equation (3), are

$$▯_1^2, ▯_2^2 = \frac{▯_1 + ▯_2}{2▯_1} + \frac{▯_2}{2▯_2} \mp \sqrt{\frac{1}{4}\left(\frac{▯_1+▯_2}{▯_1} + \frac{▯_2}{▯_2}\right)^2 - \frac{▯_1▯_2}{▯_1▯_2}} \text{ -------- (4)}$$

if, (1) ... (2)

$$\frac{(1)}{} \quad \frac{1 + 1 + 2}{} \quad \frac{}{} \qquad \text{------- (5)}$$

$$\frac{(2)}{} \quad \frac{1 + 1 + 2}{} \quad \frac{}{} \qquad \text{------- (6)}$$

General solutions of equation (1) and (2) is

$$( ) = {}^{(1)}\cos(+ \emptyset_1) + {}^{(2)}\cos(+ \emptyset_2) \text{ -------- (7)}$$

$$( ) = {}^{(1)}\cos(+ \emptyset_1) + {}^{(2)}\cos(+ \emptyset_2) \text{ ------- (8)}$$

Where, $_1^{(1)}, _1^{(2)}, \emptyset_1, \emptyset_2$ can be found using for, $m_1 = m$, $m_2 = 2m$; $k_1 = k$, $k_2 = 2k$, equation (4)

$$^2 = (2-\sqrt{3}) \quad ; \quad ^2 = (2+\sqrt{3}) \qquad \text{--------- (9)}$$

When k =1000 N/m, and m = 20 kg, $_1 = 3.6602$ rad/sec, and $_2 = 13.6603$ rad/sec

$$_1 = \frac{2}{ - {}^2 + 2} = 1.36604, \quad _2 = \frac{2}{ - {}^2 + 2} = -0.36602$$

with $(0) = 1$, $\dot{}(0) = 0$, $(0) = -1$, $\dot{}(0) = 0$,

gives, $^{(1)} = -0.36602$, $^{(2)} = -1.36603$, $\emptyset_1 = 0$, $\emptyset_2 = 0$

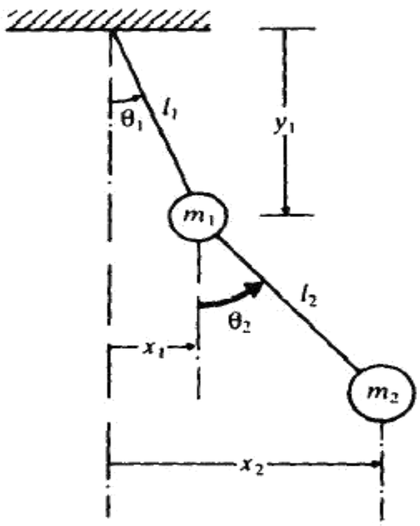
Response of the system is,

$$_1( ) = -0.36602\cos(3.6603\ ) - 1.36603\cos(13.6603\ )$$

$$_2( ) = -0.5\cos(3.6603\ ) + 0.5\cos(13.6603\ )$$

Problem 2]: Set up the differential equations of motion for the double pendulum shown in figure. Using the coordinates $_1$ and $_2$ and assuming small amplitudes. Find the natural frequencies the ratio of amplitudes, and the locations of nodes for the two modes of vibration when $_1 = _2 =$, $_1 = _2 =$.

Solution:



Taking moment about o and mass $_1$,

$$( ) + ( ) - ( ) = - + (-) \text{ --1}$$

Assuming $\cong 2$,

Similarly,

$$^2 ( ) = - \qquad \text{-------- (2)}$$

using the relations, $= \frac{}{}$, $= \frac{}{}$

equation (1) and (2) becomes,

$$^2 \frac{}{} + \frac{}{} + \frac{}{} = 0 \text{ ---- (3)}$$

and, $= 0$ -------- (4)

when,

then equation (3) and (4) becomes,

$+ 3 \quad - \quad = 0$ ------ (5)

$= 0$ ------ (6)

For harmonic motion, $; = 1, 2, 3, \ldots$

Equation (5) and (6) becomes,

$= 0$ ------ (7)

$= 0$ ------ (8)

From which the frequency equation can be obtained as;

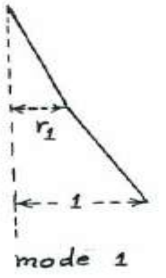
$-(4\ ^2)\ ^2+2\ ^2=0$

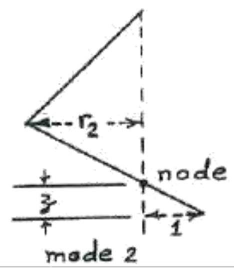
i.e., $\ ^2 = (2 \pm \sqrt{2})$

$\therefore \ _1 = 0.7654\sqrt{\ }, \quad _2 = 1.8478\sqrt{\ }$

Ratio of amplitude is given by,

In mode 1, $= 0.7654\sqrt{\ }$ $)^{(1)} = 0.4142$





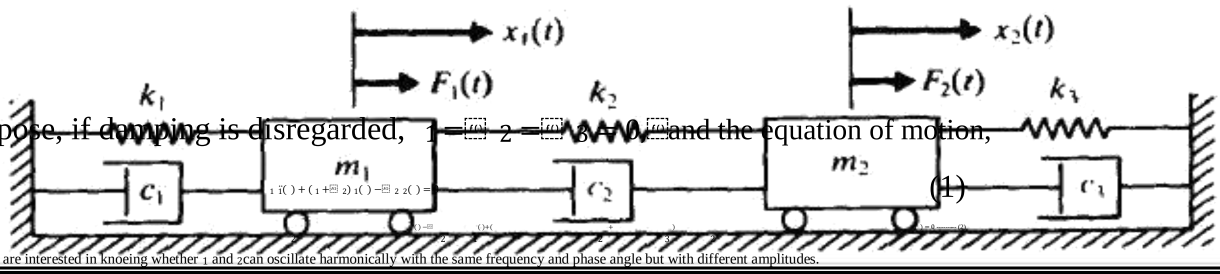
In mode 2, $_2 = 1.8478\sqrt{\ }$, $_2 = (\ ^1)^{(2)} = -2.4133$

One node located at z:

$= \dfrac{1 -}{2.4133} = 0.2930$

Free Vibration Analysis of an Undamped System:

For the free vibration analysis of the system shown in figure; we set $_1() = \ _2() = 0$.



Suppose, if damping is disregarded, $_1 = \ _2 = \ _3 = 0$, and the equation of motion,

$_1\ddot{}() + (_1 + \ _2)\ _1() - \ _2\ _2() =$ (1)

$= 0$ -------- (2)

Here, we are interested in knoeing whether $_1$ and $_2$ can oscillate harmonically with the same frequency and phase angle but with different amplitudes.

Assuming that it is possible to have harmonic motion of 1 and 2 at the same frequency and the same phase angle ∅, we take the solutions of equation (1) and (2) as

1( ) = 1cos(+ ∅) ------ (3)
2( ) = 2cos(+ ∅) ------ (4)

Where, 1 and 2 are constant that denote the maximum amplitudes of 1( ) and 2( ), and ∅ is the phase angle. Substitute equation (3), (4) into equation (1), (2), we obtain,

[{- 2 + (+ 2)} − 2 2]cos(+ ∅) = 0 ------- (5)

[{- 2 + (2 + 3)} − 2]cos(+ ∅) = 0 ------- (6)

Since equation (5), (6) must be satisfied for all values of the time 't', the terms between brackets must be zero. This yields

{- 2 + (+ 2)} − 2 2 = 0 ------ (7)

{- 2 2 + (2 + 3)} − 2 = 0 ------ (8)

Which represent two simultaneously homogeneous algebraic equations in the unknown 1 and 2.

It can be seen that equation (7), (8) are satisfied by the trivial solution 1 = 2 = 0, which implies that there is no vibration.

For a non-trivial solution of 1 and 2, the determines of the coefficient of 1 and 2 must be zero:

det[(− 2 + (+ 2)) − 2 ; − 2 (− 2 + (2 + 3))] = 0

( )− {(+ 2) 2 + (2 + 3) } 2 + {(+ 2)(2 + 3) − 2 2 } = 0 ------ (9)

equation (9) is called the frequency or characteristic equation because solution of this equation yields the frequencies or the characteristic values of the system. The roots of equation (9) are given by,

2 1 , 2 2 = 1/2 {(+ 2) 2 + (2 + 3) 1} ∓ 1/2 [{(1 + 2) 2 + (2 + 3) } 2 − 4 {(+ 2)(2 + 3) − 2 2 } / 1 2]^{1/2} ------ (10)

This shoes that it is possible for the system to have a non-trivial harmonic solution of the form of equation (3), (4) when is equal to 1 or 2 gives by equation (10), we call 1 and 2 the natural frequencies of the system.

The values of 1 and 2 remain to be determined. These values depend on the natural frequencies 1 and 2. We shall denote the values of 1 and 2 corresponding to 1 as (1) 1 and (1) 2 and those corresponding to 2 as (2) 1 and (2) 2. Further, since the equation (7), (8) are homogeneous, only the ratios 1 = { (1) 2 / (1) 1 } and 2 = { (2) 2 / (2) 1 } can be found.

For 2 = 2 1, equation (7), (8) gives

(1) 2 / (1) 1 = 1 = (− 2 1 + (+ 2)) / 2 = 2 / (− 2 1 + (2 + 3)) ------- (11)

(2) 2 / (2) 1 = 2 = (− 2 2 + (+ 2)) / 2 = 2 / (− 2 2 + (2 + 3)) ------- (12)

Torsional System

INTRODUCTION

value, always smaller than the exact value, of the fundamental natural frequency. Rayleigh s method, which is based on Rayleigh s principle, also gives an approximate value of the fundamental natural frequency, which is always larger than the exact value. Proof is given of Rayleigh s quotient and its stationariness in the neighborhood of an eigenvalue. It is also shown that the Rayleigh s quotient is never lower than the first eigen-value and never higher than the highest eigenvalue. Use of the static deflection curve in estimating the fundamental natural frequencies of beams and shafts using Rayleigh s method is presented. Holzer s method, based on a trial-and-error scheme, is presented to find the natural frequencies of undamped, damped, semidefinite, or branched translational and torsional systems. The matrix iteration method and its extensions for finding the small-est, highest, and intermediate natural frequencies are presented. A proof for the conver-gence of the method to the smallest frequency is given. Jacobi s method, which finds all the eigenvalues and eigenvectors of real symmetric matrices, is outlined. The standard eigenvalue problem is defined and the method of deriving it from the general eigenvalue problem, based on the Choleski decomposition method, is presented. Finally, the use of MATLAB in finding the eigenvalues and eigenvectors of multidegree-of-freedom systems is illustrated with several numerical examples.

Learning Objectives

After you have finished studying this chapter, you should be able to do the following:

-
-

- Find the approximate fundamental frequency of a composite system in terms of the natural frequencies of component parts using Dunkerley s formula.
- Understand Rayleigh s principle, and the properties of Rayleigh s quotient, and com-pute the fundamental natural frequency of a system using Rayleigh s method.
- Find the approximate natural frequencies of vibration and the modal vectors by using Holzer s method.
- Determine the smallest, intermediate, and highest natural frequencies of a system by using matrix iteration method and its extensions (using matrix deflation procedure). Find all the eigenvalues and eigenvectors of a multidegree-of-freedom system using Jacobi s method.
- Convert a general eigenvalue problem into a standard eigenvalue problem based on the Choleski decomposition method.
- Solve eigenvalue problems using MATLAB.

Introduction

In the preceding chapter, the natural frequencies (eigenvalues) and the natural modes (eigenvectors) of a multidegree-of-freedom system were found by setting the characteristic determinant equal to zero. Although this is an exact method, the expansion of the charac-teristic determinant and the solution of the resulting *n*th-degree polynomial equation to obtain the natural frequencies can become quite tedious for large values of *n*. Several ana-lytical and numerical methods have been developed to compute the natural frequencies and

DETERMINATION OF NATURAL FREQUENCIES AND MODE SHAPES

mode shapes of multidegree-of-freedom systems. In this chapter, we shall consider Dunkerley s formula, Rayleigh s method, Holzer s method, the matrix iteration method, and Jacobi s method. Dunkerley s formula and Rayleigh s method are useful only for esti-mating the fundamental natural frequency. Holzer s method is essentially a tabular method that can be used to find partial or full solutions to eigenvalue problems. The matrix itera-tion method finds one natural frequency at a time, usually starting from the lowest value. The method can thus be terminated after finding the required number of natural frequen-cies and mode shapes. When all the natural frequencies and mode shapes are required, Jacobi s method can be used; it finds all the eigenvalues and eigenvectors simultaneously.

Dunkerley s Formula

Dunkerley s formula gives the approximate value of the fundamental frequency of a com-posite system in terms of the natural frequencies of its component parts. It is derived by making use of the fact that the higher natural frequencies of most vibratory systems are large compared to their fundamental frequencies [7.1 7.3]. To derive Dunkerley s formula, consider a general n-degree-of-freedom system whose eigenvalues can be determined by solving the frequency equation, Eq. (6.63):

$$| -[k] + v^2[m] = 0$$

or

$$-\frac{1}{v^2}[I] + [a][m] \qquad (7.1)$$

$= 0$

For a lumped-mass system with a diagonal mass matrix, Eq. (7.1) becomes

$$-\frac{1}{\omega^2}\begin{bmatrix}1 & 0 & \cdots & 0\\ 0 & 1 & \cdots & 0\\ \vdots & & & \vdots\\ 0 & 0 & \cdots & 1\end{bmatrix}\begin{Bmatrix}U\end{Bmatrix} + \begin{bmatrix}a_{11} & a_{12} & \cdots & a_{1n}\\ a_{21} & a_{22} & \cdots & a_{2n}\\ \vdots & & & \vdots\\ a_{n1} & a_{n2} & \cdots & a_{nn}\end{bmatrix}\begin{bmatrix}m_1 & 0 & \cdots & 0\\ 0 & m_2 & \cdots & 0\\ \vdots & & & \vdots\\ 0 & 0 & \cdots & m_n\end{bmatrix}\begin{Bmatrix}U\end{Bmatrix} = 0$$

that is,

$$\begin{bmatrix}\left(-\frac{1}{\omega^2} + a_{11}m_1\right) & a_{12}m_2 & \cdots & a_{1n}m_n\\ a_{21}m_1 & \left(-\frac{1}{\omega^2} + a_{22}m_2\right) & \cdots & a_{2n}m_n\\ \vdots & \vdots & & \vdots\\ a_{n1}m_1 & a_{n2}m_2 & \cdots & \left(-\frac{1}{\omega^2} + a_{nn}m_n\right)\end{bmatrix} = 0 \quad (7.2)$$

DUNKERLEY S FORMULA

The expansion of Eq. (7.2) leads to

$$\left(\frac{1}{\omega^2}\right)^n - (a_{11}m_1 + a_{22}m_2 + \cdots + a_{nn}m_n)\left(\frac{1}{\omega^2}\right)^{n-1} + (a_{11}a_{22}m_1m_2 + a_{11}a_{33}m_1m_3 + \cdots + a_{n-1,n-1}a_{nn}m_{n-1}m_n - a_{12}a_{21}m_1m_2 - \cdots - a_{n-1,n}a_{n,n-1}m_{n-1}m_n)\left(\frac{1}{\omega^2}\right)^{n-2} - \cdots = 0 \qquad (7.3)$$

This is a polynomial equation of *n* th degree in $(1/\omega^2)$. Let the roots of Eq. (7.3) be denoted as $1/\omega_1^2, 1/\omega_2^2, \cdots, 1/\omega_n^2$. Thus

$$\left(\frac{1}{\omega^2} - \frac{1}{\omega_1^2}\right)\left(\frac{1}{\omega^2} - \frac{1}{\omega_2^2}\right)\cdots\left(\frac{1}{\omega^2} - \frac{1}{\omega_n^2}\right)$$

$$= \left(\frac{1}{\omega^2}\right)^n - \left(\frac{1}{\omega_1^2} + \frac{1}{\omega_2^2} + \cdots + \frac{1}{\omega_n^2}\right)\left(\frac{1}{\omega^2}\right)^{n-1} - \cdots = 0 \qquad (7.4)$$

Equating the coefficient of $(1/\omega^2)^{n-1}$ in Eqs. (7.4) and (7.3) gives

$$\frac{1}{\omega_1^2} + \frac{1}{\omega_2^2} + \cdots + \frac{1}{\omega_n^2} = a_{11}m_1 + a_{22}m_2 + \cdots + a_{nn}m_n \qquad (7.5)$$

$$\text{v}_1 \quad \text{v}_2 \qquad \text{n} \qquad 2$$

In most cases, the higher frequencies v2, v3, Á , vn are considerably larger than the fun-damental frequency v1, and so

$$\frac{1}{\text{v}_i^2} \;\text{V}\; \frac{1}{\text{v}_1^2}, \quad i = 2, 3, \text{Á}, n$$

Thus, Eq. (7.5) can be approximately written as

$$\frac{1}{\text{v}_1^2} \text{M}\; a_{11}m_1 + a_{22}m_2 + \text{Á} + a_{nn}m_n \tag{7.6}$$

This equation is known as *Dunkerley s formula*. The fundamental frequency given by Eq. (7.6) will always be smaller than the exact value. In some cases, it will be more conve-nient to rewrite Eq. (7.6) as

$$\frac{1}{\text{v}_1^2} \text{M} \frac{1}{\text{v}_{1n}^2} + \frac{1}{\text{v}_{2n}^2} + \text{Á} + \frac{1}{\text{v}_{nn}^2} \tag{7.7}$$

where $\text{v}_{in} = (1/a_{ii}m_i)^{1/2} = (k_{ii}/m_i)^{1/2}$ denotes the natural frequency of a single-degree-of-freedom system consisting of mass mi and spring of stiffness kii, i = 1, 2, Á , n. The use of Dunkerley s formula for finding the lowest frequency of elastic systems is presented in references [7.4, 7.5].