VARDHAMAN COLLEGE OF ENGINEERING

(Autonomous)

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S. No	UNITS
1	**BASIC PROBABILITY THEORY:** Rules for combining probability, Probability Distributions, Random variables, density and distribution functions. Mathematical expectation. Binominal distribution, Poisson distribution, normal distribution, exponential distribution, Weibull distribution.
2	**RELIABILITY**: Definition of Reliability. Significance of the terms appearing in the definition. Component reliability, Hazard rate, derivation of the reliability function in terms of the Hazarad rate, Hazard models. **FAILURES**: Causes of failures, types of failures, Modes of failure, Bath tub curve, Effect of preventive maintenance. Measures of reliability: mean time to failure and mean time between failures.
3	**CLASSIFICATION OF ENGINEERING SYSTEMS:** Series, parallel, series-parallel, parallel-series and non-series-parallel configurations. Expressions for the reliability of the basic configurations. **RELIABILITY LOGIC DIAGRAMS:** Reliability evaluation of Non-series-parallel configurations: minimal tie-set, minimal cut-set and decomposition methods. Deduction of the minimal cut sets from the minimal path sets.
4	**DISCRETE MARKOV CHAINS:** General modeling concepts, stochastic transitional probability matrix, time dependent probability evaluation and limiting state probability evaluation. Absorbing states. **CONTINUOUS MARKOV PROCESSES:** Modeling concepts, State space diagrams, Stochastic Transitional Probability Matrix, Evaluating limiting state Probabilities. Reliability evaluation of repairable systems.
5	**SERIES SYSTEMS AND PARALLEL SYSTEM:** Series systems, parallel systems with two and more than two components, Network reduction techniques. Minimal cut set/failure mode approach

UNIT-I

Basic Concepts on Probability

Probability means whether a certain event has a good chance of occurring or not.
Its value lies between 0 and 1.

Rules for combining probabilities

1) ***Independent Events***:
Two events are said to be independent if the occurrence of one event does not affect the probability of occurrence of the other event.
Example: Throwing a dice and tossing coin are independent events.

2) Mutually exclusive events:
Two events are said to be mutually exclusive or disjoint if they cannot happen at the same time.
Example: (i) When throwing a single die, the events 1, 2, 3, 4, 5 and 6 spots are all mutually exclusive because two or more cannot occur simultaneously
(ii) Similarly success and failure of a device are mutually exclusive events since they cannot occur simultaneously.

3) Complimentary Events:
Two outcomes of an event are said to be complementary, if when one outcome does not occur, the other must occur.
If the outcomes A & B have probabilities P(A) and P(B), then
$P(A) + P(B) = 1$ $P(B) = P(\overline{A})$

Example: When tossing a coin, the outcomes head and tail are complementary since
$P(head) + P(tail) = 1$ or
$P(head) = P(\overline{tail})$
$P(tail) = P(\overline{head})$
Therefore we can say that two events that are complementary events are mutually exclusive also. But the converse is not necessarily true i.e, two mutually exclusive events are not necessarily complementary.

4) ***Conditional Events***;
Conditional events are events which occur conditionally on the occurrence of another event or events.
Consider two events A & B and also consider the probability of event A occurring under the condition that event B has occurred.

$$P(A/B) = \frac{P(A\cap B)}{P(B)}$$

5) Simultaneous occurrence of events:
Occurrence of both A & B - Mathematically it is represented as $A\cap B$, A AND B, AB.

Case (i) Independent, then the probability of occurrence of each event is not influenced by the probability of occurrence of the other.

$P(A/B) = P(A)$
$P(A \cap B) = P(A) – P(B)$
And $P(B/A)- = P(B)$
Case (ii) Events are dependent
If two events are not independent, then the probability of occurrence of one event is influenced by the probability of occurrence of the other

Therefore, $P(A \cap B) = P(B/A).P(A)$
$= (PA/B).P(B)$

Numerical Problem - 1
An engineer selects two components A & B. The probability that component A is good is 0.9 & the probability that component B is good is 0.95.What is the probability of both components being good.

$P(A \text{ good} \cap B \text{ good}) = P(A \text{ good}) (B \text{ good})$
$= 0.9 \times 0.95 = 0.80$

Numerical Problem - 2
One card is drawn from a standard pack of 52 playing cards. Let A be the event that it is a red card and B be the event that it is a face card. What is the probability that both A & B occur.

$P(A) = 26/52$
Given that ‘A’ has occurred
Then the sample space for B is 26 states, out of which 6 are those of a face card.
Therefore, $P(B/A = 6/26$
$P(A \cap B) = 6/26 \times 26/52 = 6/52$
$P(A \cap B) = P(B/A)\ P(A)$

6) ***Occurrence of atleast one of two events***:
The occurrence of atleast one of two events A and B is the occurrence of A or B or BOTH. Mathematically it is the union of the two events and is expressed as (AUB), (A or B) or (A x B)

Case (i) – Events are independent but not mutually exclusive.
$P(AUB) = P(A \text{ OR } B \text{ OR BOTH } A \text{ AND } B)$
$= 1 – P(\text{NOT } A \text{ AND NOT } B)$
$= 1 – P(\bar{A} \cap \bar{B})$
$= 1 – P(\bar{A})\ .\ P(\bar{B})$
$= 1 – (1-P(A))\ (1-P(B))$
$= P(A) + P(B) – P(A).P(B)$

Using Venn diagram
$P(AUB) = P(A) + P(B) – P(A \cap B)$
If $P(A) = 0.9$ and $P(B) = 0.95$
$P(AUB) = P(A) + P(B) – P(A).P(B)$
$= 0.9 + 0.95 – 0.9 \times 0.95 = 0.995$

Case (ii) – Events are independent and mutually exclusive In the case of events A & B being mutually exclusive, then the probability of their simultaneous occurrence P(A).P(B) must be zero by definition.

P(AUB) = P(A) + P(B)

Case (iii) – Events are not independent
If two events are not independent then
$P(A \cup B) = P(A) + P(B) - P(A \cap B)$
$= P(A) + P(B) - P(B/A).P(A)$
$= P(A) + P(B) - P(A/B).P(B)$

Numerical Problem - 3
A cinema hall gets electric power from a generator run by a diesel engine. On any give day, the probability that the generator is down (event A) is 0.025 and the probability that the diesel engine is done (event B) is 0.04. What is the probability that the cinema house will have power on any given day.

Assume that the occurrence of events A & B are independent of each other.
Probability that the
Cinema hall does not have power given by the probability of the event that either the diesel engine or generator is down.
$Q = P_r(A \cup B) = P(A) + P(B) - P(A)\,P(B)$
$= 0.025 + 0.04 - 0.025 \times 0.04 = 0.064$

Therefore, the probability that the cinema house have power
= R = 1- 0.064 = 0.936

Numerical Problem - 4
In a sample of 60 mails, 10 of them contains only defective heads, five contain only defective tails and and five contain both the defects. What is the probability that a mail that is selected randomly contains either defective head or a defective tail?

Let X denote the event that a mail contains a defective head and Y denote the event containing a defective tail.
Than $P(X) = \frac{10+5}{60} = 0.25$
$P(Y) = \frac{5+5}{60} = 0.1667$
$P(X \cap Y) = 5/60 = 0.0833$

The probability that a mail contains either of the two defects is $P(X \cup Y) = P(X) + P(Y) - P(X \cap Y) = 0.334$

Therefore, probability that a mail contains no defect is 40/60 = 0.6667 = 1-P(XUY)

Random Variables

To study about a system's behavior for the application of probability theory to reliability evaluation, a series of experiments must be performed or a data collection scheme should be deduced.
To apply the probability theory to occurrence of these values or events which are random in nature, we need to study these variables called as Random Variables.

∴ Random variable is a variable quantity which denotes the result or outcome of a given random experiment.
A random variable is one that can have only a discrete number of states or countable values.

A random variable can be either "discrete" or "continuous".

A discrete random variable is one that can have only a discrete number of states or countable values.
Ex: 1.Tossing a coin - Outcomes are heads or tails.
2. Rolling a dice - Outcomes are 1,2,3,4,5 or 6.

A continuous random variable is one which takes an infinite number of values or if its range forms a continuous set of real numbers. This does not mean that the range extends from $-\infty$ to $+\infty$. It only means that there are infinite number of possibilities of the value.

Ex: 1. The life time of a light bulb.
2. If electric current have values between 5A and 10 A, then it indicates a continuous random
variable.

Probability Density Function

The probabilities associated with the random variables can be described by a formula called Probability density function or Probability mass function.

We use the notation f(x) for the probability density function.

Ex : 1. Consider the throw of a dice
Let the random variable associated with the outcome be 'X'.
The value of X are 1, 2, 3, 4, 5 and 6.
$f(1) = P(x=1) = 1/6$
$f(2) = P(x=2) = 1/6$
$\therefore f(1) = f(2) = \ldots\ldots\ldots\ldots = f(6) = 1/6$
$f(x) = 1/6$ = Constant density function.

2. Consider the rolling of two dice. What is f(x) for the sum of dots facing up?
X= Total Sum of dots
$P(x) = f(x) = ?$

$P(x=2) = f(2)$

∴ $f(x) = \frac{x-1}{36}$ for x= 2,3,4,5,6,7

∴ $f(x) = \frac{13-x}{36}$ for x= 8,9,10,11,12

Probability Distribution Function

If ‘x’ is a random variable, then for any real number x, the probability that ‘x’ will assume a value less than or equal to x is called Probability distribution functions.
It is indicated as F(x)

f(x) = P(x)

$F(x) = P(X \leq x)$

Ex: Consider the rolling of a single dice.
f(x) = 1/6

f(1) = f(2) = ………..f(6) = 1/6

$F(1) = P(X \leq 1) = f(1) = 1/6$

$F(2) = P(X \leq 2) = f(1) + f(2)$
$= 1/6 + 1/6 = 2/6$

$F(3) = P(X \leq 3) = f(1) + f(2) + f(3)$
$= 1/6 + 1/6 + 1/6 = 3/6$

F(4) = 4/6

F(5) = 5/6

F(6) = 6/6 = 1

Suppose a random variable X has the following density function.

X	0	1	2	3	4	5
f(x)	1/32	5/32	10/32	10/32	5/32	1/32

Then, the Probability distribution function is given by

X	0	1	2	3	4	5
F(x)	1/32	1/32 + 5/32 =6/32	6/32 + 10/32 =16/32	16/32 + 10/32 =26/32	26/32 + 5/32 =31/32	31/32 + 1/32 =32/32

Relation between Probability density function and distribution function:

$F(x) = \sum f(x)$ (Discrete Random variable)

F(x) = $\int f(x)dx$ (Continuous Random variable)

A random variable 'x' and the corresponding distribution function F(x) are said to be continuous if the following condition is satisfied for all 'x'.

f(x) = $\frac{d}{dx}\ f(x)$

Mathematical Expectation

It is useful to describe the random behavior of a system by one or more parameters rather than as a distribution. This is particularly useful in the case of system reliability evaluation.

This parametric description can be achieved using numbers known mathematically as moments of distribution.
The most important of these moments is the expected value, which is also referred to as average mean value.

Mathematically it is the first moment of the distribution.

Consider a Probability model with outcome x_1, x_2, x_3……x_n and the probability of each is P_1, P_2, P_3………….P_n. then the expected value of the variable is E(x) = P_1x_1 + P_2X_2+ P_3X_3+……….P_nX_n = $\sum_{i=1}^{n} x_1 P_1$

Expected value E(x) of a discrete random variable x having 'n' outcomes x_i each with a probability of occurrence P_i is E(x) = $\sum_{i=1}^{n} x_i P_i$ where $\sum_{i}^{n} P_1 = 1$

In case of continuous random variable, the equation can be modified from the summation to integration.

E(x) = $\int x\ f(x)dx$

Expected value is the weighted mean of the possible value using their Probability of occurrence as the weighing factor.

Variance and Standard Deviation

The expected value is the most important distribution parameters in reliability evaluation. But to know the amount of 'spread' or 'dispersion' of a distribution, the second moment of distribution. i.e., variane V(x) should be deduced.

The variane of a random variable 'x' is defined as the expectation of the square of deviation of 'x' from E(x).

m = E(x) = $\int x\ f(x)dx$
Variance = $\sigma^2 = \int (x-m)^2\ f(x)dx$
The quantity $'\sigma'$ is called standard Deviation.
The K^{th} moment of a random variable 'x' about its expectation is defined as $M_k = E[x - E(x)]^K$.
The second moment of distribution is known as variance V(x) (K=2)

$$\begin{aligned} V(x) &= E[x - E(x)]^2 \\ &= E[x^2 - 2x\,E(x) + E^2(x)] \\ &= E(x^2) - E(2x\,E(x)) + E[\,E^2(x)] \\ &= E(x^2) - 2\,E(x)\,E(x) + E^2(x) \end{aligned}$$

$= E(x^2) - 2E^2(x) + E^2(x)$

$= E(x^2) - E^2(x)$

$= \sum_{i=1}^{n} x_i^2 P_i - E^2(x)$

Properties of the binomial distribution

The binomial distribution can be represented by the general expression:

$$(p+q)^n$$

For the expression to be applicable, four specific conditions are required. These are:

(a) There must be a fixed number of trials, i.e. n is known
(b) Each trial must result in either a success or a failure, i.e., only two outcomes are possible and p + q = 1.
(c) All trials must have identical probabilities of success and therefore of failure, i.e., the values of p and q remain constant, and
(d) All trials must be independent (this property follows from (c) since the probabilities of success in trial i must be constant and not affected by the outcome of trials 1, 2, ..., (i-1)).

In order to apply the binomial distribution and to evaluate the outcomes and their probability of occurrence of a given experiment or set of trials, the expression $(p+q)^n$ must be expanded into the form of equations and

$$(p+q)^n = p^n + np^{n-1}q + \frac{n(n-1)}{2!}p^{n-2}q^2 + \ldots + \frac{n(n-1)\ldots(n-r+1)}{r!}p^{n-r}q^r + \ldots . + q^n$$

If equation is compared with, it is seen that the coefficient of the (r+1)th term in the binomial expansion represents the number of ways, i.e., combinations, in which exactly r failures and therefore (n-r) successes can occur in n trials and is equal to ${}_nC_r$. Therefore each coefficient in equation can be directly evaluated from the definition of ${}_nC_r$ as discussed and the probability of exactly r successes or (n-r) failures in n trails can be evaluated from

$$p_r = \frac{n!}{r!\,(n-r)!}p^r q^{n-r}$$

$$= {}_nC_r p^r q^{n-r}$$

$$= {}_nC_r p^r (1-p)^{n-r}$$

Substituting of equations gives

$$(p+q)^n = \sum_{r=0}^{n} c_r p^r q^{n-r} = 1$$

Numerical example – I

A coin is tossed 5 times. Evaluate the probability of each possible outcome and draw the probability mass (density) function and the probability distribution function.

Solution

In this example n=5, p=q=1/2. Using the binomial expansion the outcomes, the probability of exactly r heads or (n-r) tails and the cumulative probability are determined as shown in below table.

Number of		Individual probability		Cumulative
heads	Tails	Expression	Value	probability
0	5	$_5C_0(1/2)^o(1/2)^5$	1/32	1/32
1	4	$_5C_1(1/2)^1(1/2)^4$	5/32	6/32
2	3	$_5C_2(1/2)^2(1/2)^3$	10/32	16/32
3	2	$_5C_3(1/2)^3(1/2)^2$	10/32	26/32
4	1	$_5C_4(1/2)^4(1/2)^1$	5/32	31/32
5	0	$_5C_5(1/2)^5(1/2)^0$	$\underline{1/32}$	32/32
			$\sum = 1$	

In the above table the values of individual probability have been summated and a value of unity obtained.

The results are plotted as a probability mass (density) function and probability distribution functions in figures.

The probability density function is symmetrical. This only occurs when p=q=1/2 since in this case the success and failure events can be interchanged without any alteration in the numerical value of any of the individual outcomes. This will not be the case when p and q are unequal.

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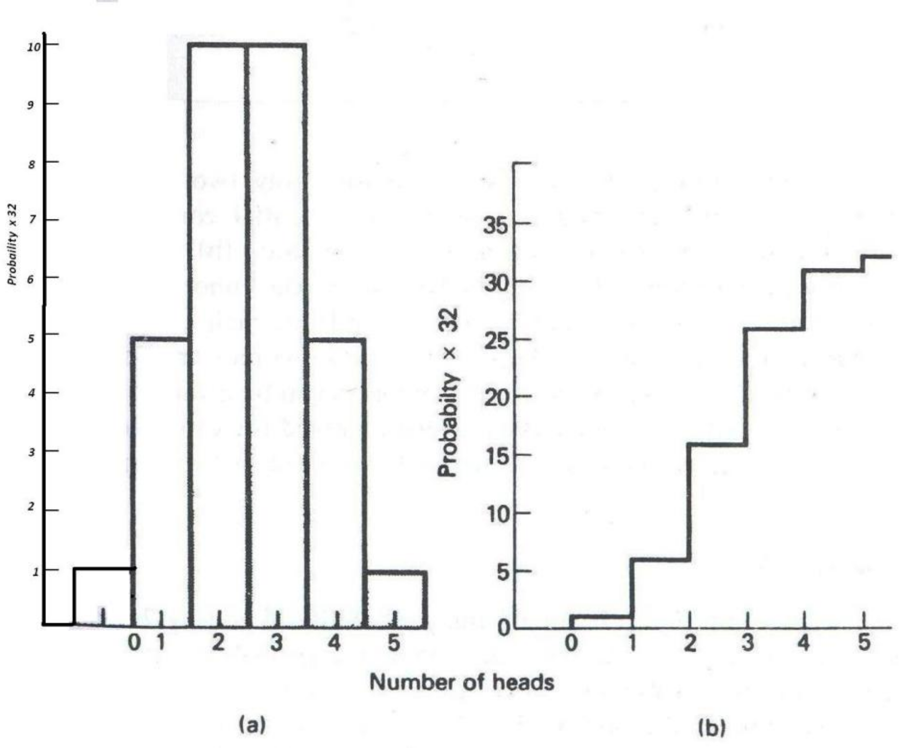


Results for Example (a) Probability density (mass) function. (b) Probability distribution function

Numerical example – II

Consider the case in which the probability of success in a single trial is ¼ and four trials are to be made. Evaluate the individual and cumulative probabilities of success in this case and draw the two respective probability functions.

Solution

$n=4, p=1/4, q=3/4$

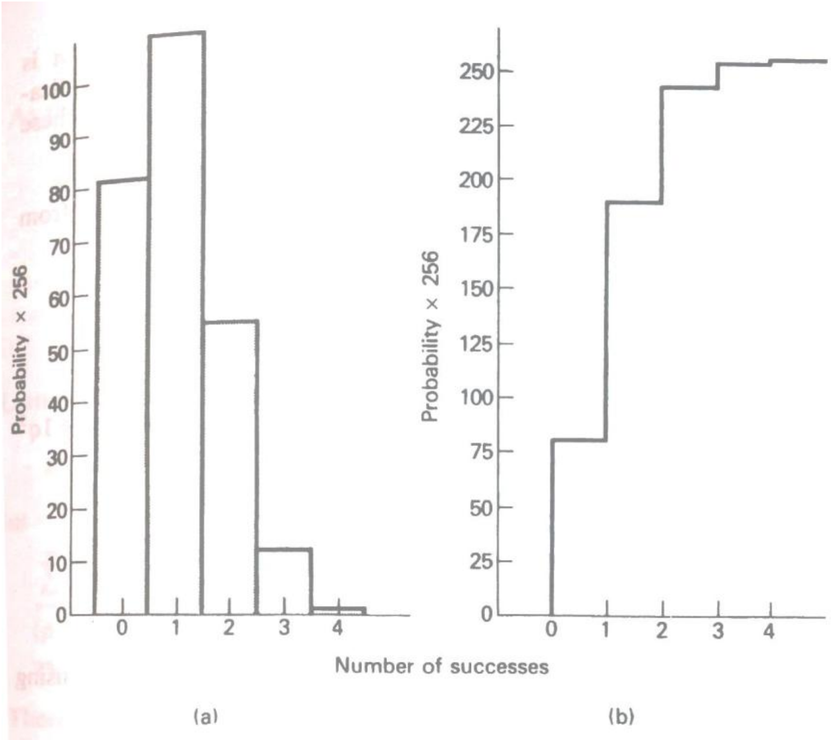
Number of successes	Number of Failures	Individual probability	Cumulative probability
0	4	$(3/4)^4 = 81/256$	81/256
1	3	$4(1/4)(3/4)^3 = 108/256$	189/256
2	2	$6(1/4)^2(3/4)^2 = 54/256$	243/256
3	1	$4(1/4)^3(3/4) = 12/256$	255/256
4	0	$(1/4)^4 = 1/256$	256/256
		$\sum = 1$	

Numerical example – III

A die is thrown in 6 times. Evaluate the probability of getting 2 spots on the upper face 0, 1, 2, . . . , 6 times and draw the probability mass (density) function and the probability distribution function.

Solution:

On each throw, the probability of getting 2 spots on the upper face is 1/6 and the probability of not getting 2 spots is 5/6. If these two events are defined as success and failure respectively, then, although there are six possible outcomes on each throw, the problem has been constrained to have two outcomes and the binomial distribution becomes applicable.



(a) (b)

Consequently n=6, p=1/6 and q=5/6. The probability results are shown in below table.

Number of successes	Individual probability	Cumulative probability
0	$(5/6)^6$ = 15625/46656	15625/46656
1	6(1/6) $(5/6)^5$ = 18750/46656	34375/46656
2	$15(1/6)^2(5/6)^4$ = 9375/46656	43750/46656
3	$20(1/6)^3(5/6)^3$ = 2500/46656	46250/46656
4	$15(1/6)^4(5/6)^2$ = 375/46656	46625/46656
5	$6(1/6)^5(5/6)$ = 30/46656	46655/46656
6	$(1/6)^6$ = 1/46656	46656/46656
	$\sum = 1$	

Expected value and Standard Deviation for Binomial Distribution:

The two most important parameters of a distribution are the expected or mean value and the standard deviations.
The binomial distribution is a discrete random variable and therefore the expected value and standard deviation can be evaluated using equation.

E(x) = $\sum_{i=1}^{n} x_i P_i$ where $\sum_{i}^{n} P_i = 1$

=$\sum_{x=0}^{n} x\,(n_{c_x}\, p^x q^{n-x})$ [P(r,n) = $n_{c_r}\, p^r (1-p)^{n-r})$]

= $\sum_{x=0}^{n} x\, \frac{n!}{x!(n-x)!} p^x q^{n-x}$

As the contribution to this summation made by x=0 is zero, then

E(x) = $\sum_{x=1}^{n} \frac{n!}{(x-1)!(n-x)!}\, p^x q^{n-x}$

E(x) = $\sum_{x=1}^{n} \frac{xn(n-1)!}{x(x-1)!(n-x)!}\, p\, p^{x-1} q^{n-x}$

= $\sum_{x=1}^{n} \frac{n(n-1)!}{(x-1)!(n-x)!}\, p\, p^{x-1} q^{n-x}$

= np $\sum_{x=1}^{n} \frac{(n-1)!}{(x-1)!(n-x)!}\, p^{x-1} q^{n-x}$

Let n-1 = m and x-1 = y

$E(x) = np \sum_{y=0}^{m} \frac{(n-1)!}{y!(n-1-y)!} p^y q^{n-1-y}$

$E(x) = np \sum_{y=0}^{m} \frac{m!}{y!(m-y)!} p^y q^{m-y}$

$Since, \sum_{y=0}^{m} \frac{m!}{y!(m-y)!} p^y q^{m-y} = 1$

E(x) = np

Expected value or Mean value and Standard Deviation for Exponential Distribution:

The expected value of a continuous random variable having a range (0 , ∝) is given by

$E(x) = \int_0^{\propto} t.f(t)\, dt$

$E(x) = \int_0^{\propto} t.\lambda\, e^{-\lambda t}\, dt$

This can be integrate by parts

Let $u = t$ and $v = -e^{-\lambda t}$

$du = dt$ $dv = \lambda\, e^{-\lambda t}\, dt$

$\therefore E(t) = \int udv = [uv]_0^{\propto} - \int_0^{\propto} vdu$

$= [-t\, e^{-\lambda t}]_0^{\propto} - \int_0^{\propto} e^{-\lambda t}\, dt$

$= \left[-t\, e^{-\lambda t}\right]_0^{\propto} - \left[\frac{1}{\lambda} e^{-\lambda t}\right]_0^{\propto}$

$= 0 + \frac{1}{\lambda}$

$\therefore E(t) = \frac{1}{\lambda}$

$$\sigma^2 = \int_0^{\propto} t^2.\lambda\, e^{-\lambda t}\, dt - E^2(t)$$

$u = t^2$ $v = -e^{-\lambda t}$

$du = 2tdt$ $dv = \lambda\, e^{-\lambda t}\, dt$

$$\sigma^2 = \int udv - E^2(t)$$

Integrating by parts

$= [uv]_0^{\propto} - \int_0^{\propto} vdu - E^2(t)$

$= [-t^2\, e^{-\lambda t}]_0^{\propto} - \int_0^{\propto} -2t\, e^{-\lambda t}\, dt - E^2(t)$

$\sigma^2 = 0 + \frac{2}{\lambda}.\frac{1}{\lambda} - \frac{1}{\lambda^2} = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}$

$$\sigma^2 = \frac{1}{\lambda^2}$$

$$\sigma = \frac{1}{\lambda}$$

∴ Expected value and Standard deviation of an Exponential Distribution are equal.

Mean Time To Failure (MTTF)

The expected value of a failure density function is often designated as the mean time to failure MTTF.

In case of exponential distribution this is equal to the reciprocal of the failure rate λ.

E(t) $= \int_o^p t\, f(t) dt$

$f(t) = -\frac{d}{dt} R(t)$

MTTF $= -\int_o^p t\, dR(t)$

$= \int_o^p R(t) dt$

$= [-t\, R(t)]_o^p + \int_o^p R(t) dt$

MTTF $= \int_o^p R(t) dt$

Reliability Analysis of series networks using exponential distribution

Let $R_1(t)$, $R_2(t)$ . . . . $R_n(t)$ be the reliabilities of 'n' components connected in series.

$R_s(t) = R_1(t)\, R_2(t) \,.\,.\, R_n(t)$

$= \prod_{i=1}^{n} R_i t$

Let $R_1(t) = e^{-\lambda_1 t}$ & $R_2(t) = e^{-\lambda_2 t}$ ...

$R_n(t) = e^{-\lambda_n t}$

∴ $R_s(t) = e^{-(\lambda_1 \lambda_2 \lambda_3 \ldots\ldots + \lambda_n)t}$

$= e^{-\sum_{i=1}^{n} \lambda_i t}$

Total failure rate

$$\lambda_s(t) = \sum_{i=1}^{n} \lambda_i t$$

Therefore, Hazard rate function for the system is determined by summing the hazard rate function of the 'n' independent components.

MTTF $= \int_o^p R_s(t) dt = \int_o^p e^{-\sum_{i=1}^{n} \lambda_i t}\, dt$

MTTF $= \frac{1}{\lambda} = \frac{1}{\sum_{i=1}^{n} \lambda_i} = \frac{1}{\sum_{i=1}^{n} \frac{1}{MTTF\ _i}}$

If all the components connected in series have the same failure rates

$\lambda_1 = \lambda_2 = \lambda_3 \ldots\ldots \lambda$ then $\lambda = n\lambda_1$

MTTF $= \frac{1}{n\lambda 1}$

Parallel configuration

Let $R_1(t)$ $R_2(t)$ . . . $R_n(t)$ be the reliabilities of the components connected in parallel.

$R_s(t) = 1 - [(1-R_1(t))\,(1-R_2(t))\ .\ .\ .\ .\ .\ (1-R_n(t))]$

$= 1- \prod_{i=1}^{n}[1 - R_i(t)]$

$Rs(t) = 1-\prod_{i=1}^{n}\left[1 - e^{-\lambda_i t}\right]$

Where λ_i = failure rate of its component

Two components in parallel

$R_s(t) = 1- \left[\left(1 - e^{-\lambda_1(t)}\right)\left(1 - e^{-\lambda_2(t)}\right)\right]$

$= e^{-\lambda_1 t} + e^{-\lambda_2 t} - e^{-(\lambda_1+\lambda_2)t}$

$\text{MTTF} = \int_o^P R_s(t) = \int_o^P e^{-\lambda_1 t}dt + \int_o^P e^{-\lambda_2 t}dt - \int_o^p e^{-(\lambda_1+\lambda_2)t}$

$= \frac{1}{\lambda_1} + \frac{1}{\lambda_2} - \frac{1}{\lambda_1+\lambda_2}$

If $\lambda_1 = \lambda_2 = \lambda$

Then $R_s(t) = 2e^{-\lambda t} - e^{-2\lambda t}$

$\text{MTTF} = \frac{2}{\lambda} - \frac{1}{2\lambda}$

Numerical example 1

An aircraft engine consists of three modules having constant failure rates of $\lambda_1 = 0.002$ $\lambda_2 = 0.015$ and $\lambda_3 = 0.0025$ failures per operating hour. What is the reliability function for the engine and what is the MTTF?

Solution:

$R(t) = e^{-(0.002+0.015+0.0025)t}$

$= e^{-0.0195t}$

$\text{MTTF} = \frac{1}{0.0195} = 51.28\ operating\ hours$

Numerical example 2

Consider a four component system of which the components are independent and identically distributed with Constant Failure Rate (CFR). If $R_s(100) = 0.95$, find the individual component MTTF?

Solution:

$R_s(100) = e^{-100\lambda_s} = e^{-100(4x\lambda)} = 0.95$

$\lambda = \frac{-\ln(0.95)}{400} = 0.000128$

MTTF = $\frac{1}{0.000128} = 7812.5$

Numerical example 3

A Simple electronic circuit consists of 6 transistors each having a failure rate of 10^{-6} f/hr, 4 diodes each having a failure rate of 0.5×10^{-6}f/hr, 3 capacitor each having a failure rate of 0.2×10^{-6}f/hr, 10 resistors each having a failure rate of 5×10^{-6}f/hr. Assuming connectors and wiring are 100% reliable (these can be included if considered significant), evaluate the equivalent failure rate of the system and the probability of the system surviving 1000hr if all components must operate for system success.

Solution

λ_s = Equivalent failure rate of the system

$= 6 \text{ X } (1 \text{ X } 10^{-6}) + 4 \text{ X } (0.5 \text{ X } 10^{6}) + 3 \text{ X } (0.2 \text{ X } 10^{-6}) + 10 \text{ X}(5 \text{ X } 10^{-6}) + 2 \text{ X } (2 \text{ X } 10^{-6})$

$= 6.26 \times 10^{-5}$ f/hr

$R_s(t) = e^{-\lambda_s t}$

$R_s(1000) = \exp(-6.26 \text{ X } 10^{-5} \times 1000)$

$= 0.9393$

Since $Q_s(t) = 1 - R_s(t)$

$Q_s(1000) = 1 - 0.9393$

$= 0.0707$

Poisson distribution

Poisson distribution is a discrete probability distribution that expresses the probability of a given number of events occurring in a fixed interval of time and/or space if these events occur with a known average rate and independently of the time.

The Poisson distribution can also be used for the number of events in other specified intervals such as distance, area or volume.

Poisson distribution is an approximation to binomial distribution. It is used for large values of n and small p

$$P(x, \lambda) = \frac{e^{-\lambda} \lambda^x}{x!}$$

λ = shape parameter which indicates the **average** number of events in the given time interval.

= Mean value

Numerical example - 1

A rare disease has an incidence of 1 in 1000 person-years. Assuming that members of the population are affected independently, find the probability of ‘k’ cases in a population of 10,000 for k=0, 1, 2

Solution:

Expected mean $\lambda = 0.001 \text{ X } 10{,}000$

$= 10$

$P(x = 0) = \frac{e^{-10} 10^0}{0!} = 0.0000454$

$P(x = 1) = \frac{e^{-10} 10^1}{1!}$ = 0.000454

$P(x = 2) = \frac{e^{-10} 10^2}{2!}$ = 0.00227

Numerical example - 2

In a large system the average number of cable faults per year per 100 km of cable is 0.5. Consider a specified piece of cable 10km long and evaluate the probabilities of 0, 1, 2 etc, faults occurring in (a) a 20 year period, and (b) a 40 year period

Solution:

Assuming the average failure rate data to be valid for the 10km cable and for the two periods being considered, the expected failure rate λ is,

$$\lambda = \frac{0.5\ x\ 10}{100} = 0.05\ f/yr$$

(a) For a 20 year period,
E(x) = 0.05 X 20 = 1.0

And

$P_x = \frac{1.0^x e^{-1.0}}{x!}$ for x = 0, 1, 2,…

(b) For a 40 year period,
E(x) = 0.05 X 40 = 2.0

And

$P_x = \frac{2.0^x e^{-2.0}}{x!}$ for x = 0, 1, 2,…

UNIT-II

Reliability : The reliability of a device is considered high if it had repeatedly performed its function with success and low if it had tended to fail in repeated trials.

The reliability of a system is defined as the probability of performing the intended function over a given period of time under specified operating conditions.

The above definition can be broken do on into four parts:

i) Probability
ii) Intended function

iii) Given period of time
iv) Specified operating conditions

Probability: Because, the reliability is a Probability, the reliability of system R_j is governed by the equation $0 \leq R_s \leq 1$. The equality sign hold good in case of equipment called one shot equipment.

Intended function: It is also defined to as the successful operation.

Example-1: As an example, let us consider the building up of voltage by a dc shunt generator. For some reasons, let us assume that the voltage is not build up.
We say that the dc shunt generator has failed to do its job. The failure in this contest doesn't imply any physical failure, but only the operational failure.

Example-2: Lightning arrester : The lightning arrestor should burst in the event of occurrence of a lightning stroke. On the occurrence of a lightning stroke, if the lightning arrestor bursts, there is the physical failure or damage but operationally it is successful.
On the other hand, if it doesn't burst there is no physical failure, but yet there is an operational failure and we say that the lightning arrestor has failed (operationally).

Given period of time: Any component has some useful life period, within which time the component should operate successfully. For example, a power transformer has a useful life of at least 20 to 25 years.
If, it fails within this time period, then the instrument is said to be unreliable and if it fails after its useful life period, then we say it is reliable.

Specified operating or environmental conditions: Any equipment is supposed to perform its duty satisfactorily under contain specified operating condition such as temperature, humidity, pressure and altitude.
Though an equipment is able to perform its duty satisfactorily in a cold country yet it may fail when used under hot climatic conditions.

Component Reliability

It is usual for a large system to be divided into components for the purpose of reliability evaluation. A component is that part of a system which is treated as a single entity for the purpose of reliability evaluation. There is no clear distinction between component and system. The same unit can be considered as component (or) system depending on circumstances. For instance, a generating unit is considered as a component while dealing with the reliability of entire power system. Same can be treated as a complex system consisting of several components like the boiler, Turbine and Generator, etc.

Components can be classified into two groups – Non-repairable components and Repairable components.
Non-repairable components are components that cannot be repaired or the repair is uneconomical.
Repairable components are components which can be repaired upon failure and thus their life histories consist of alternating operating and repair periods.
In the reliability evaluation of power systems, it is the repairable type that is of greater interest.

For several reasons, a component put into service fails after sometime, called the TIME TO FAILURE (T), this can be recognized as a random variable and the reliability of a component at any time can be expressed as

$$\begin{aligned} R(t) &= P(T>t) \\ &= 1 - F(t) \\ &= 1 - P(T \leq t) \end{aligned}$$

Reliability Function:

f(x) = Probability density function

F)x) = Probability distribution

All components have a different failure rate, hence these time-to-failure obey a probability distribution, thus probability value is a function of time that is specified or considered.

f(t)= density function which indicate the rate of failures per hour.

R(t)= Reliability function

An additional function which is one of the most extensively used function in reliability evaluation is the hazard rate h(t).

In terms of failure the hazard rate is a measure of the rate at which failures occur or the instantaneous failures/hour.

$$f(t) = \frac{\text{no of failures}}{\text{no of component x operating hours}}$$

$$h(t) = \frac{\text{no of failures}}{\text{no of component at the begining of internal x operating hours}}$$

Thus the hazard rate is dependent on the number of failures in a given time period and the number of components exposed to failures.

PROBLEMS:

1. The field test data in respect of 172 components is as given below. Calculate failure density rate and hazard rate

Time internal hrs	0-1000	1000-2000	2000-3000	3000-4000	4000-5000	5000-6000
Failure in internal	59	24	29	30	17	13

Solution :

f(t)	**h(t)**
59/172 X 1000	59/172X1000
24/172X100	24/113x1000
29/172X100	29/89x1000
30/172X100	30/60X100
17/172X100	17/30X100
13/172X100	13/13X100

2. The component failure data for ten components subjected to a life test are given below. Find the failure density rate and hazard rate.

Failure	1	2	3	4	5	6	7	8	9	10
Operating time hrs	8	20	34	46	63	86	111	141	186	206

Solution :

Time internal	0-8	8-20	20-34	34-46	46-63	63-86	86-111	111-141	141-186	186-206
f(t)	$\frac{1}{10x8} = 0.0125$	$\frac{1}{10x12}$	$\frac{1}{10x14}$	$\frac{1}{10x12}$	$\frac{1}{10x17}$	$\frac{1}{10x23}$	$\frac{1}{10x25}$	$\frac{1}{10x20}$	$\frac{1}{10x45}$	$\frac{1}{10x20}$

h(t)	$\frac{1}{10x8}$ =0.0125	$\frac{1}{9x12}$	$\frac{1}{8x14}$	$\frac{1}{7x12}$	$\frac{1}{6x17}$	$\frac{1}{5x23}$	$\frac{1}{4x25}$	$\frac{1}{3x20}$	$\frac{1}{2x45}$	$\frac{1}{1x20}$

HAZARD FUNCTION : h(t)

The probability function of the random variable ‘T’ can be determined by

$$f(t) = \lim_{\Delta t\to 0} \frac{P(\, t<T \le t+\Delta t)}{\Delta t} \qquad \Delta t = \text{increment time}$$

Suppose at t=0, N(0) components are put to work. At any time ‘t’ suppose the number of components is N(t),

then $f(t) = \frac{N(t)-N(t+\Delta t)}{N(0).\Delta t}$

If the probability in the above case is calculated as conditional probability condition being that the component should be working at t, then the function is Hazard function h(t).

$$h(t) = \lim_{\Delta t\to 0} \frac{P(\, t<T \le t+\Delta t)}{\Delta t} \Big/ \text{at } (T > t)$$

$$h(t) = \frac{N(t)-N(t+\Delta t)}{N(t).\Delta t}$$

General Reliability functions

All components have a different failure rate, hence these times to failure obey a probability distribution. This Probability value is a function of time that is specified or considered.

Let f(t) = failure density function which indicates the rate of failures per hour

R(t) = Reliability function

An additional function which is one of the most extensively used in reliability evaluation is the hazard rate h(t).
In terms of failure, the hazard rate is a measure of the rate at which failures occur. It indicates the instantaneous failures / hour.

$$f(t) = \frac{\text{Number of failures}}{\text{Number of components} \;\; \text{X operating hours}}$$

$$h(t) = \frac{\text{Number of failures}}{\text{Number of components at the begining of interval X operating hours}}$$

The hazard rate is dependent on the number of failures in a given time period and the number of components exposed to failure.

Failure density function f(t) is the rate of failures per hour.
Hazard rate h(t) is the instantaneous failures per hour.

Derivation of Reliability function R(t) in terms of hazard rate h(t)

A non - repairable component is of use only till the failure occurs and if the component fails, we have to replace it with a new component

Such a component is described by its life time T, a random variable.
Since 'R' is a function of 't' (operating time), the reliability can be defined as

$$R(t) = P(T>t) \quad \text{---------} (1)$$

$$= 1 - P(T \leq t)$$

But $P(T \leq t) = F(t) =$ failure distribution function

$$\therefore R(t) = 1 - F(t) \text{ -------}(2)$$

$$\text{Failure density function } f(t) = \frac{d\,F(t)}{dt} = -\frac{d\,R(t)}{dt} \text{ --------}(3)$$

Consider a case in which fixed number N_0 of identical component are tested.

Let $N_s(t)$ = Number of components surviving at time 't'
$N_f(t)$ = Number of components failed at time 't'
$N_s(t) + N_f(t) = N_0$

At any time 't' the reliability function

$$R(t) = \frac{N_s(t)}{N_o} \text{ -------}(4)$$

$$= \frac{N_o - N_f(t)}{N_o} = 1 - \frac{N_f(t)}{N_o} \text{ ------ } (5)$$

Similarly the probability of failure or cumulative failure distribution

$$F(t) = \frac{N_f(t)}{N_o} \text{ -------}(6)$$

From equations 5 and 6 we get equation 2

$$R(t) = 1 - F(t)$$

$$\frac{d\,R(t)}{dt} = -\frac{d\,F(t)}{dt} = -\frac{1}{N_o}\frac{d\,N_f(t)}{dt} \text{ ------- } (7)$$

$$f(t) = -\frac{d\,R(t)}{dt}$$

$$= \frac{1}{N_o}\frac{d\,N_f(t)}{dt} \text{ -------- } (8)$$

The failure density function and hazard rate are identical only at t= 0.

∴ The general expression for hazard rate at time 't' is

$$h(t) = \frac{1}{N_s(t)}\frac{d\,N_f(t)}{dt} \text{ --------- } (9)$$

$$= \frac{N_o}{N_o}\cdot\frac{1}{N_s(t)}\frac{d\,N_f(t)}{dt}$$

$$= \frac{N_o}{N_s(t)}\,\frac{1}{N_o}\frac{d\,N_f(t)}{dt}$$

$$= \frac{1}{R(t)}\,f(t) = \frac{f(t)}{R(t)}$$

$$\therefore\ h(t) = \frac{f(t)}{R(t\ )} \text{ --------- } (10)$$

From equation 8 $\quad f(t) = \frac{-d}{dt}R(t)$

$$h(t) = \frac{-1}{R(t)}\frac{d\,R(t)}{dt} \text{ --------- } (11)$$

Let us consider

$$\frac{d}{dt}[\ln R(t)] = \frac{1}{R(t)}\frac{d\,R(t)}{dt}$$

$$= \frac{1}{R(t)}\frac{d}{dt}[1 - F(t)]$$

$$= \frac{1}{R(t)} - \frac{d\,F(t)}{dt} = -\frac{f(t)}{R(t)} = -h(t)$$

$$\therefore \frac{d}{dt}\ln R(t) = -h(t)$$

$$\ln R(t) = -\int_0^t h(t)\,dt$$

$$R(t) = e^{-\int_0^t h(t)\,dt} \text{ -------}(12)$$

For a constant hazard rate, h(t) = λ= number of areas

$$\therefore R(t) = e^{-\int_0^t \lambda\,dt} = e^{-\lambda t}$$

$$R(t) = e^{-\lambda t} \text{ --------- } (13)$$

Relation between R(t), Q(t), F(t) , f(t) and h(t)

R(t) = Reliability function Q(t) = Unreliability function h(t) = hazard rate function

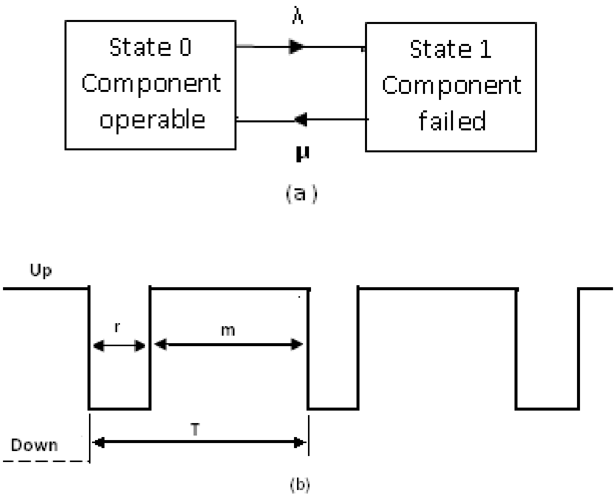
F(t) = failure Distribution function

f(t) = failure density function which indicates the rate of failures per hour

	R(t)	**Q(t) = F(t)**	**f(t)**	**h(t)**
R(t)	–	1- Q(t)	$-\int f(t)\,dt$	$e^{-\int_0^t h(t)\,dt}$
Q(t)	1- R(t)	–	$\int f(t)\,dt$	$1-e^{-\int_0^t h(t)\,dt}$
f(t)	$-\frac{d\,R(t)}{dt}$	$\frac{d\,Q(t)}{dt}$	–	$h(t)e^{-\int_0^t h(t)\,dt}$
h(t)	$-\frac{d}{dt}\ln R(t)$	$\frac{\frac{d\,Q(t)}{dt}}{1-Q(t)}$	$\frac{f(t)}{-\int f(t)dt}$	–

Measures of Reliability:

Consider a single repairable component for which the failure rate and repair rate are constant. The state transition diagram for this component is shown below.



Single component system (a) State space diagram (b) Mean time/ state diagram

Let, λ = failure rate of the component
μ = repair rate of the component
m = mean operation time of the component
r = mean repair time of the component

The two system states and their associated transitions can be shown chronologically on a time graph. The mean values of up and down times can be used to give the average performance of this two state system. This is shown in figure b.

In figure b, the period T is the system cycle time and is equal to the sum of the mean time to failure (MTTF) and mean time to repair (MTTF). This cycle time is defined as the mean time between failures (MTBF). Some times, MTBF is used in place of MTTF. It is evident however that there is a significant conceptual difference between MTTF and MTBF. The numerical difference between them will depend on the value of MTTR. In practice the repair time is usually very small compared with the operating time and therefore the numerical values of MTTF and MTBF are usually very similar.

The following relationships can therefore be defined

$$m = \text{MTTF} = 1/\lambda \qquad r = \text{MTTR} = 1/\mu$$

$$T = \text{MTBF} = m+r = 1/f$$

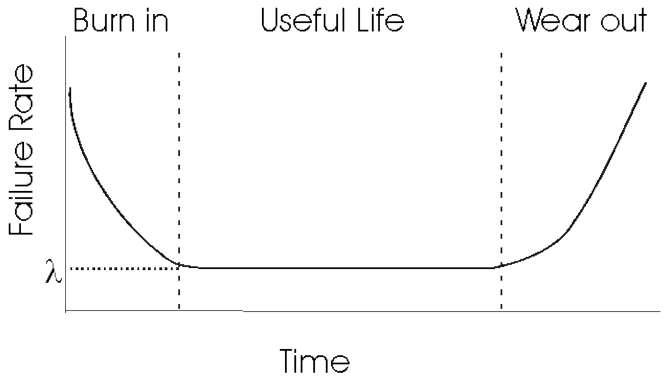
Where f = cycle frequency, i.e., the frequency of encountering a system state.
The failure rate λ is the reciprocal of the mean time to failure, MTTF, with the times to failure counted from the moment the component begins to operate to the moment it fails. Similarly, the repair rate μ is the reciprocal of the mean time to repair, MTTR, with these times counted from the moment the component fails to the moment it is returned to an operable condition.

$$\lambda = \frac{\text{number of failures of a component in the given period of time}}{\text{total period of time the component was operating}}$$

$$\mu = \frac{\text{number of repairs of a component in the given period of time}}{\text{total period of time the component was repaired}}$$

Bath – Tub Curve

The plot of hazard rate versus time is referred to as Bath – tub curve**.** Most of the components have a high failure density rate at the beginning and the failure rate decreases with time. The failures in the beginning are mainly due to defects in design and due to the improper manufacturing techniques, etc.



- ➢ These will be detected and corrected so that failure rate or hazard rate decreases with time. This is indicated by the portion $0 - t_1$ of the curve.
- ➢ Between t_1 and t_2 the hazard rate is more or less constant and beyond t_2 the hazard rate increases with time due to normal wear and tear.

- The period 0- t_1 is referred to as **debugging period** or **burn – in** period and the failures are refined to as infant mortality.
- The period t_1 – t_2 is called as the **useful life** period. In this period the failures are chance failures or random failures.
- The period beyond t_2 is called the **wear-out period** and the failures are mainly due to aging effect. These failures are called wear-out failures.
- Because of the shape of the curve, it is called as Bath- Tub- Curve.
- The Bath – Tub – Curve can be divided into three regions namely

i) Decreasing hazard rate region
ii) Constant hazard rate region
iii) Increasing hazard rate region

UNIT-III

The reliability evaluation of engineering systems can be obtained by drawing RLD Reliability Logic Diagram or Reliability Block Diagram RBD or RLG Reliability Logic Graph.

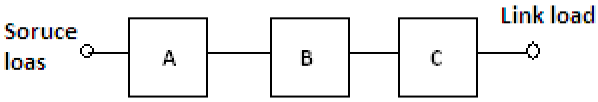
RLD or RBD : In a RBD each of the engineering system components is indicated by a block. The RLD shows the components in order that the intended function can be accomplished. It may or may not have any resemblance to the actual arrangement of the components.

Ex:- Let us consider a hypothetical generating station consisting of three generators A, B, and C rated at 5Mw, 4Mw and 3Mw respectively to feed a load connected at the station bus- bars.

Case 1: System load is 10Mw

The generating station is successful if a power to 10Mw can be supplied. This requires that (A) and, (B) and (C) must be in an operating condition.

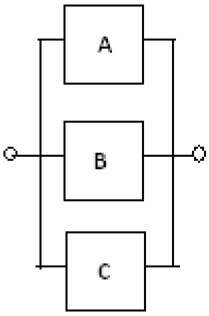
Each of the generators is represented by a block and they are all connected in series. The AND is reflected as a series connection both the components concerned. The RLD is as shown in figure



Case 2: System load is 2MW

A power of 2mw can be applied by generator (A)or (B) or (C)

This OR is reflected as a parallel connection among the components in the RLD is as shown in the figure



Case 3: Required power 6 MW.

This power requirement can be met by the power functioning of generators. { (A) 'AND' (B) } . OR. {(B) . AND. (C)}, OR. {(C) AND. (A)}

From an examination of the RLD's we can observe the following:

1. The RLD may or may not have resemblance to the physical arrangement of the components.
2. One and the same energy system will have different RLD based on different criteria for the success of the system.

For example,in the series RLD A, B, C may represent a DC shunt generator where A- Armature winding
B- Field winding and C- Commutator
The same diagram may represent the RLD of a ball where A= Banal B= Riffle and C=nib of the riffle

Reliability Logic Diagram (RLD) or Reliability Block Diagram (RBD):

The RBD shows the logical interconnections among the various components, rather than the actual connections, it may or may not have any resemblance to the physical arrangement of the components. In the RBD each component is represented by a block, whereas in the RLG each component is represented by a branch (edge) connected between two nodes, in addition to two nodes, viz., the source node(s/in) and sink node (t/out).

Once the RLD is drawn, the actual system loses its significance. One and the same system may have different RLD's based on the different success criteria and conversely, one and the same RLD may correspond to different system.

Classification of Engineering Systems:

All other parameters remaining the same, the reliability of a system depends on how the system is to be modeled. As for as the reliability evaluation is concerned, Engineering systems are classified based on their Reliability Logic Diagram as.

i) Series system (configuration).
ii) Parallel system (configuration).
iii) Series-parallel system (configuration).
iv) Parallel- series system (configuration).
v) K out of m: Good systems
v) Non-series-parallel system (configuration) or Complex System.

Series Configuration:

If a system is successful, i.e., if it is able to perform its intended function, if and only if each and every one of the n components is operative, then the system is to be modeled as a series configuration as shown in figure.

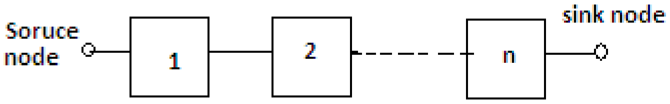


Fig: Series Configuration

The system reliability

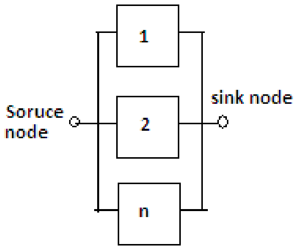
$$R_S(t) = \prod_{j=1}^{n} r_j(t) \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

Where $r_j(t)$ is the reliability of the component at the j^{th} stage (or) subsystem. The reliability of a system, which is modeled as a series configuration, is less than the least among the component reliabilities, since the reliability of each component is less than unity and the system reliability is the product the component reliabilities.

A series system is referred to as n-out of n: Good System or equivalently 1-out of n: Failed system.

Parallel Configuration:

If the success of a system is ensured by the operating condition of any one of the n components, then the system is modeled as a parallel configuration as shown in fig.



Parallel Configuration

Since the system is successful even if one of the n components is operative, the system fails if and only if each and every one of the n components fails, i.e., the system unreliability is

$Q_S(t) = \prod_{j=1}^{n} q_j(t)$……………..(2)

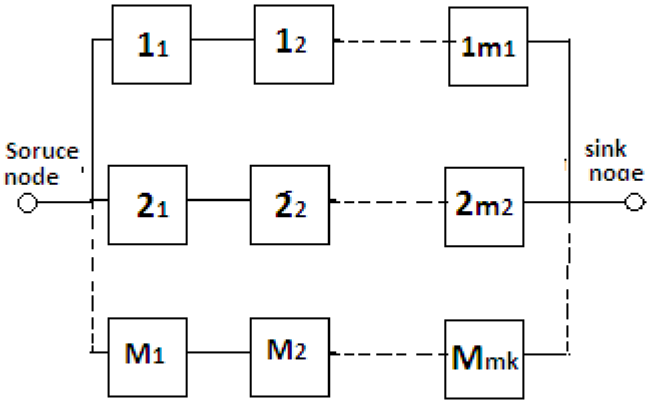
Where $q_j(t) = 1 - r_j(t)$ and the system reliability, $R_S(t) = 1 - Q_S(t)$

The unreliability of a parallel system is less than the least among the component unreliabilities or the reliability is higher than the highest among the component reliabilities.

A parallel configuration is referred to as 1-out of n: Good system (or) equivalently, n-out of n: Failed system.

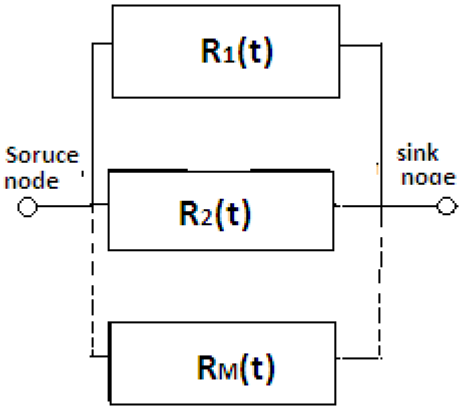
Series- Parallel configuration:

If the intended function can be achieved by different alternatives and if each of the alternatives requires the functioning of all the components connected in series in that alternative, the system is to be modeled as a series –parallel configuration as shown in fig.



Series- Parallel Configuration

Let $R_k(t)$ be the reliability of the K^{th} alternative (of the series- parallel configuration) obtained by using equation then the system can be reduced the one shown in fig.

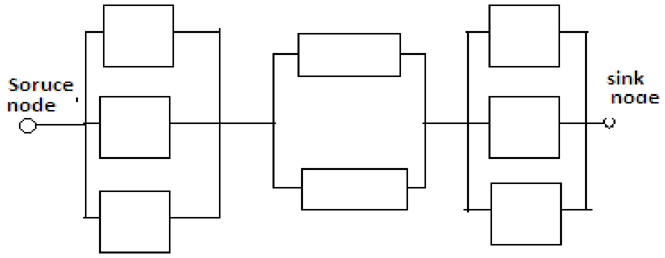


Equivalent Parallel System

The reliability of the system shown in fig can be obtained by making us of equation (1) by replacing $q_j(t)$ by $Q_j(t) = (1-R_j(t)$.

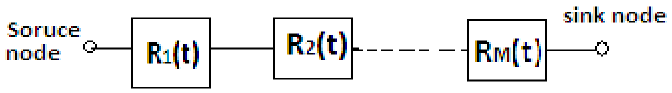
Parallel –Series Configuration:

If the function of any subsystem can be achieved by one or more components connected in parallel and if all of the subsystems must be operative to ensure the success of the system, then it is to be modeled as a parallel-series configuration as shown in fig.



Parallel-Series System

The system is successful if each subsystem has at least one operative component. The reliability is evaluated by replacing each subsystem by a single unit of equivalent reliability as shown in fig.



Equivalent series system

Where $R_S(t) = \prod_{j=1}^{n} R_j(t)$………..(3)

Where the various Rj's obtained by making use of equation(2).

r out of n: Good systems:

If the component of a system are arranged such that the system works if any 'r' components out of 'n' components present in the system are working .

Let the reliabilities of all the components be equal to R.

$R_r = P(r) = n_{C_r}\, R^r (1-R)^{n-r}$

$R_{r+1} = P(r+1) = n_{C_{r+1}}\, R^{r+1} (1-R)^{n-r-1}$

.

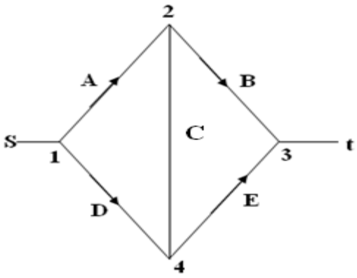
.

$R_n = P(n) = n_{C_n}\ R^n(1-R)^0$

Total reliability ‘r’ out of n = P(r) + P(r+1)+ ……P(n)
(if r=n then it is a series system and if r=1 then it is a parallel system)

Non-series-parallel system (configuration) or Complex System:

If the RBD does not fall under any one of the four basic configuration, viz. series, parallel, series-Parallel and parallel-series, then the system is to be modeled as a complex configuration. The bridge type configuration shown in fig. is an example of a complex configuration. The presence of the element C renders the system to be complex.

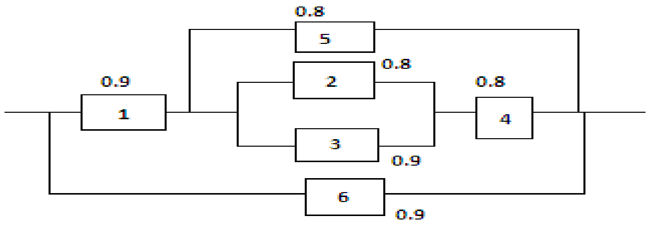


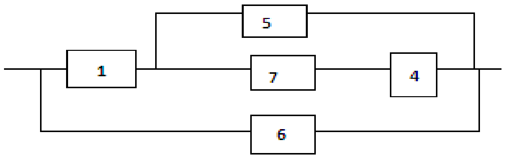
Complex Configuration

Problems:

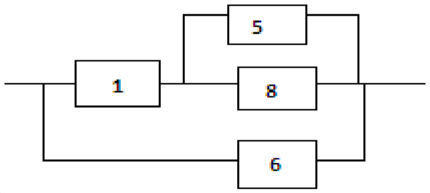
1. Calculate the reliability of the system shown using network reduction technique?

R_s= R_s

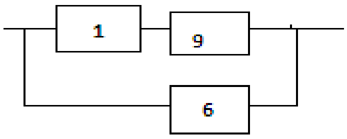




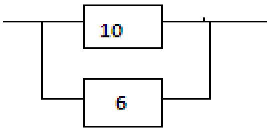
R_7= 1-{(1-0.8)(1-0.9)} = 1-0.02 =0.98



$R_8 = R_7R_4$ =0.98x0.8 =0.784



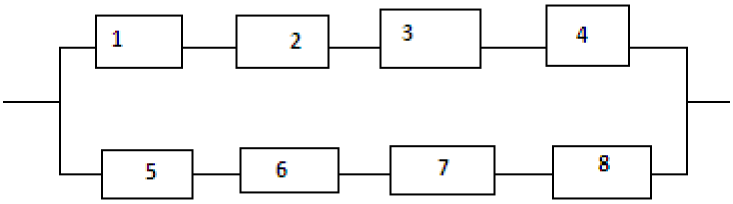
R_9= 1-{(1-R_5)(1-R_8) = 1- (0.2x 0.216) = 1-0.0432 = 0.9568

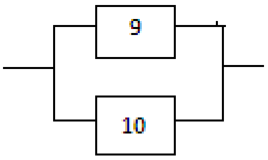


$R_{10}=R_1R_9$=0.9x0.9568 = 0.86112

R_{11}1- {(1-0.86112)(1-0.9)} =1-(0.13888x0.1) =1-0.01388 = 0.98612

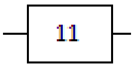
2. A system consists of 10 identical components, all of which must work for system success. What is the system reliability it each component has a reliability of 0.95.
   R_s =0.95^{-10} = 0.5987
3. A two component series system contains identical components each having a reliability of 0.99. Evaluate the unreliability of the system.
   $R_s(0.99)^2$ $Q_s = 1-(0.99)^2 =0.0199$
4. A system design required 200 identical components in series. It the overall reliabilitys must not be less than 0.99, what is the minimum reliabilitys of each component.
   R^{200}= 0.99
   R= $0.99^{1/200}$ =0.99995
5. A system consists of from components in parallel having reliabilities of 0.99, 0.95, 0.98 & 0.97. what is the reliability and unreliability of the system.
   Q_p= (1-0.99)(1-0.95)(1-098)(1-0.97) = $3x10^{-7}$
   R_p =0.9999997
6. A system is to be designed with an overall reliability of 0.99 using components having individual reliabilities of 0.7. what is the minimum number of components that must be connected in parallel.
   $Q_p = (Q_i)^n$
   $(1-0.999) =(1-0.7)^n$
   $0.01 = (0.3)^n$
   N=5.75 n=6
7. A series system has 10 identical components. It the overall system reliability must be at least 0.99, what is the minimum reliability reqired of each component
   $(R)^{10} = 0.99$ $R = (0.99)^{1/10}$
8. A parallel system has 10 identical component. If the overall system reliability must be at least 0.99, how power can these components be
   $(1-0.99) = (1-R)^{10}$
   $(1-R) = (1-0.99)^{1/10}$ =0.3690
9. A parallel system has identical components having a reliability of 0.5. what is the minimum number of component if the system reliability must be at least 0.99.
10. Derive a general expression for the reliability of the model shown below and hence evaluate the system reliability if all components have a reliability of 0.9.
    $R_{11} = R_s$





$R_9 =R_1R_2 R_3R_4$

$R_{10} = R_5R_6R_7R_8$



$R_{11} = 1 - Q_{11}$

$R_{11} = 1 - Q_9Q_{10}$

$= 1 - \{(1-R_9)(1-R_{10})\} = R_9 + R_{10} - R_9R_{10}$

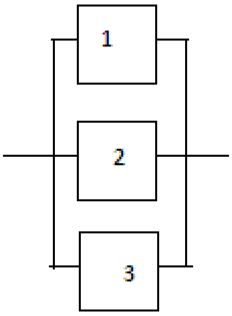
$= R_1R_2R_3R_4 + R_5R_6R_7R_8 - R_1R_2R_3R_4R_5R_6R_7R_8$

$R_1 = R_1R_2R_3R_4R_5R_6R_7R_8 = 0.9$

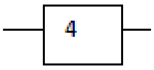
$R_{11} = 0.9^4 + 0.9^4 - 0.9^8 = 0.8817$

$R_{11} = R_s = 0.8817$

1. Find the reliability of the system below if at lest 2 should for system such and if the reliability of each component is 0.8?



R_4 is evaluated by applying the binomial distribution to component 1,2, and 3. r=2, n = 3



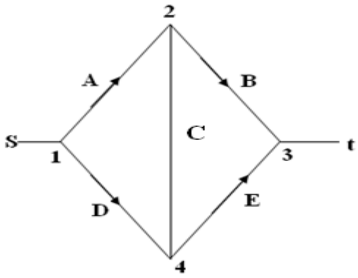
$$=P_2 = 3_{C_2}(0.8)^2(0.2) = 3X(0.8)^2(0.2)$$

$$P_3 = 3_{C_3} = (0.8)^3$$

$$P_r = P_2 + P_3 = (0.8)^3 + 3(0.8)^2(0.2)$$

Reliability Evaluation of Non-series-parallel system or Complex System

If the Reliability block diagram does not fall under any of the four basic configuration, viz. series, parallel, series-Parallel and parallel-series, then the system is to be modeled as a complex configuration. The bridge type configuration shown in fig. is an example of a complex configuration. The presence of the element C renders the system to be complex.



Complex Configuration

Reliability Evaluation of Complex Systems

Though the reliability of a complex system can be determined by various methods such as the exhaustive search for the successful states, direct canonical expansion, probability map method, probability calculus method etc., one of the following methods is commonly used:

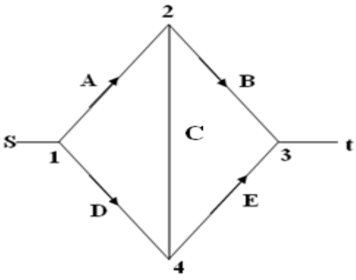
- Minimal path set (or) Minimal tie set method
- Minimal cut set method
- Decomposition method

Minimal Pathset Method:

A path is a sequence of the branches of the reliability block diagram from the input node to the output node such that the succeeding node of any branch is the same as the preceeding node of the next branch. A minimal path is one which satisfies the above property.

Accordingly, a minimal pathset may be defined as the set of the elements the proper functioning of which ensures the success of the system and no proper subset of which can ensure the success of the system. With reference to the RLG, the set of the elements which ensures the connectivity between the nodes 'S' and 't' and no subset of which [other than the set itself] ensures this connectivity is a minimal pathset. In the case of small systems the minimal pathsets can be obtained by inspection. Simple algorithms are available for the determination of the minimal pathsets of large systems, involving more number of branches. Thus, the minimal path sets (tie sets) of a bridge network are

T_1 = {AB}



T_2 = {DE}

T_3 = {ACE}

T_4 = {DCB}

A structure function X is such that

$X = \{T_1 U\ T_2 U\ T_3 U\ T_4 U\}$

The reliability of the configuration is probability of X, i.e., $R_s(t) = Pr(X)$

The symbolic reliability expression of the system is

$R_s(t) = ab + de + ace + bcd - abde - abce - abcd - acde - bcde + 2\,(abcde)$,

Where the lower case letters represent the reliabilities of the corresponding elements represented by upper case letters, i.e., $a = R_A(t)$ etc. The reliability expression thus obtained is the un-complemented symbolic reliability expression. The numerical value of the reliability is obtained by substituting the values of the component reliabilities.

Thus,, if the reliabilities of the components A, B, C, D and E are, respectively 0.70, 0.75, 0.80, 0.85 and 0.90, the overall system reliability is 0.922575.

Minimal Cutset Method:

A Cutset is a set of branches which when cut will not allow any path from input node to output node. A minimal cutest is a set of branches which satisfies this property, but no subset of this has the same property.

A minimal cutest is defined as the set of the elements, the failure of which i.e, the inoperative condition of which causes the failure of the system, and no proper subset of which can cause the failure of the system.
With reference to the RLG, a minimal cutest destroys the connectivity between the source and the sink nodes.
Thus, referring to the minimal cutsets are
$C_1 = \{\bar{A}\,\bar{D}\}$ $C_2 = \{\bar{B}\,\bar{E}\}$ $C_3 = \{\bar{A}\,\bar{C}\bar{E}\}$ $C_4 = \{\bar{B}\bar{C}\,\bar{D}\}$
The structure function $\bar{X} = \{C_1 U\ C_2 U\ C_3 U\ C_4\ \}$

The unreliability of the system is $Q_s(t) = P(\bar{X})$ and the reliability, $R_s(t) = 1 - Q_s(t)$. The symbolic unreliability of the system can be obtained as $Q_s(t) = a'd' + b'e' + a'c'e' + b'c'd' - a'c'd'e' - a'b'c'd' - a'b'c'e' - b'c'd'e' + 2(a'b'c'd'e')$, where $a' = (1-a)$ etc.
Using the same values of the component reliabilities given in the minimal pathset approach, the unreliabilities of the components A through E are 0.30, 0.25, 0.20, 0.15 and 0.1, respectively. The overall system unreliability is 0.077425 and the overall system reliability is 0.922575, as in the previous case.

While the minimal pathsets can be obtained in a straight-forward manner, there is no simple algorithm by which the minimal cut sets can be obtained. The minimal cutsets are deduced from the minimal path sets by making use of DeMorgan's laws.

While the reliability can be obtained from a knowledge of minimal pathsets, the unreliability is obtained from a knowledge of the minimal cutsets and hence the reliability is obtained. While it cannot be generalized that the number of minimal cutsets is less than the number of minimal pathsets, for most of the complex configurations the number of minimal cutsets is less than the number of minimal pathsets. For example, the number of minimal pathsets for the configuration, whereas the number of minimal cutsets is only 6.

To evaluate the reliability by the minimal tie set method the number of terms to be evaluated is 2^7-1 ie., 127. On the other hand the number of terms to be considered is only 2^6- 1 i.e., 63, if the minimal cutest method were to be used.

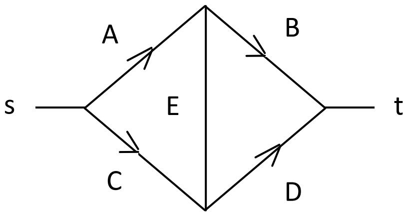
Decomposition Method

The reliability of a complex system can be evaluated by considering the combinations that renders the system a simple series-parallel or a parallel-series configuration.

In this method, one of the elements preferably the one that renders the system complex is chosen as the *corner stone* or *key element*.

If K is the keystone element, then the reliability of the system

R_s= (Reliability of modified system/given K is good) (Reliability of K) + (Reliability of modified system / given K is failed) (unreliability of K)



For the RLD shown, the presence of element E renders the system complex and E is a bidirectional element. Thus to evaluate the reliability of this system, we consider two states which are mutually exclusive with respect to E namely E is operative and E is failed.

Let us choose 'E' as the corner stone element, so that the reliability of system = R_1 e + R_2 (1-e)

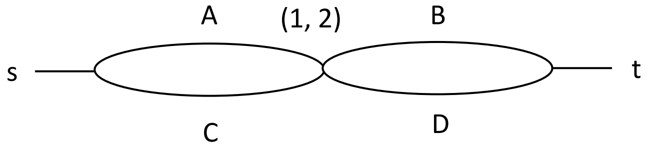
where R_1 = Reliability of modified system with E good

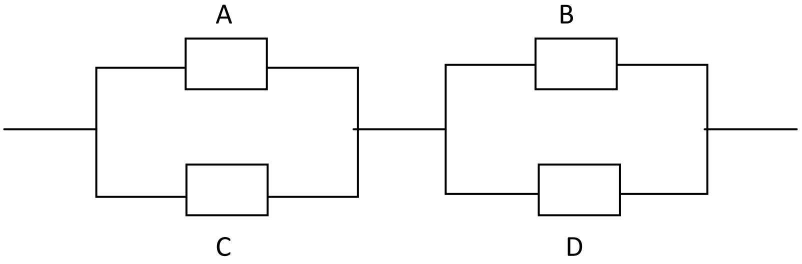
R_2 = reliability of modified system with E failed.

$R_s = R_1e + R_2(1\text{-}e)$ where 'e' is the reliability of component E.

When E is working - Calculation of R_1

When E is operative, the two nodes 1 & 2 are always connected together or they can be merged with each other to form a combined node (1, 2). The RLD can be redrawn as below

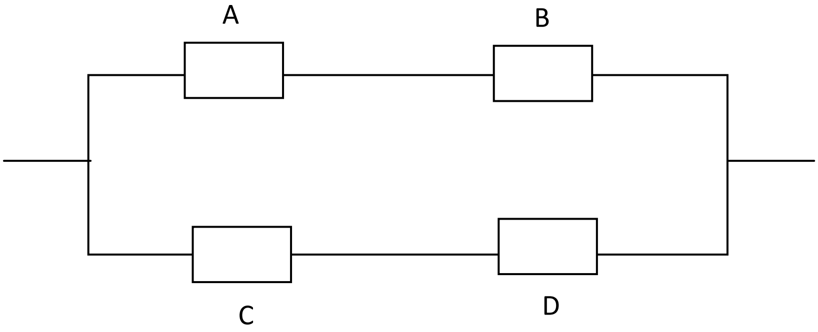




∴ Reliability of system R_1 = (a+c-ac) (b+d-bd)

When E is not - working - Calculation of R_2

'E' is failed state implies an open circuit between nodes 1 & 2 and the RLD is modified as below.



R_2 = 1-[(1-ab) (1-cd)] = 1-[1-ab-cd+abcd]
=ab+cd-abcd

∴ Overall Reliability = $R_s = R_1e + R_2(1\text{-}e)$
R_s = [(a+c-ac)(b+d-bd)]e + [(ab+cd-abcd)(1-e)]
= (ab+ad-abd+cd+cb-bcd-abc-acd+abcd)e + (ab+cd-abcd-abe-cde+abcde)
= abe+ade-abde+cde+cbe-bcde-abce-acde+abcde+ab+cd-abcd-abe-cde+abcde

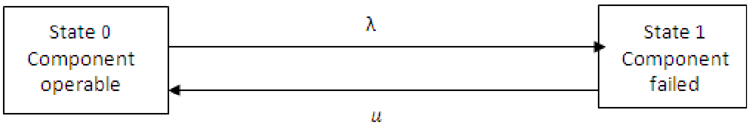
R_s = ab + cd + ade +bce –abde – bcde – abce – acde – abcd + 2abcde

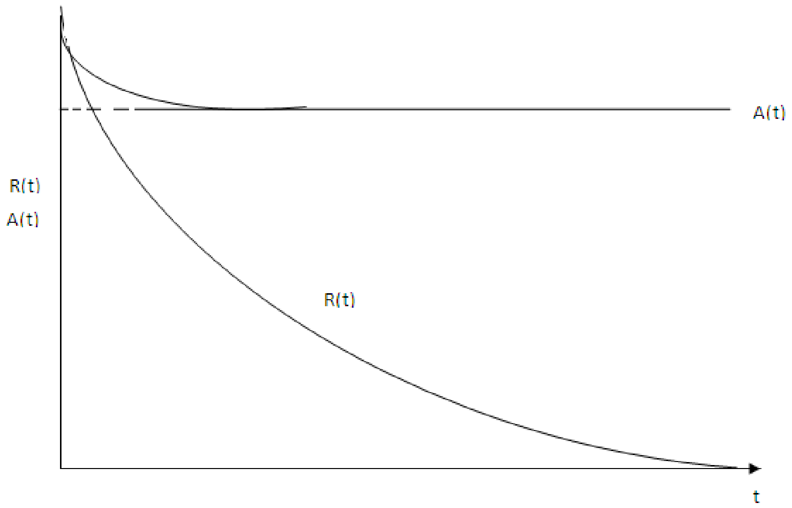
Unit – IV

MARKOV PROCESS

Transition rate concepts

Consider the case of a single repairable component for which the failure rate and repair rate are constant, i.e. they are characterized by the exponential distribution. The state transition diagram for this component is shown in below figure





Let $P_0(t)$ = probability that the component is operable at time t
$P_1(t)$ = probability that the component is failed at time t
λ = failure rate
μ = repair rate

The failure density function for a component with a constant hazard rate of was given in equation as

$$f(t) = \lambda e^{-\lambda t}$$

The density functions for the operating and failed states of the system shown in above figure are therefore

$f_o(t) = \lambda e^{-\lambda t}$ and $f_1(t) = \mu e^{-\mu t}$, respectively

The parameters λ and μ are referred to as state transition rates since they represent the rate at which the system transits from one state of the system to another.

The failure rate λ was found to be the reciprocal of the mean time to failure, MTTF, with the times to failure counted from the moment the component begins to operate to the moment it fails. Similarly, the repair rate μ is the reciprocal of the mean time to repair, MTTR, with these times counted from the moment the component fails to the moment it is returned to an operable condition. The correct interpretation of state residence time is an important point, as the failure and repair rates are sometimes incorrectly evaluated by counting the number of failures or repairs in a given period of time, and dividing by the elapsed time. The correct time value to use in the denominator is the portion of time in which the component was in the state being considered. This is less than the actual elapsed period of time unless no transitions occurred from the state. Consequently

$$\lambda = \frac{\text{number of failures of a component in the given period of time}}{\text{total period of time the component was operating}}$$

$$\mu = \frac{\text{number of repairs of a component in the given period of}}{}$$

$$\frac{\text{time}}{\text{total period of time the component was being repaired}}$$

This concept of a transition rate leads to the definition
Transition rate = number of times a transition occurs from a given state/time spent in that state

Evaluating time dependent probabilities

The relevant state space diagram for the simple single component is shown in above figure. The transitions in these diagrams were represented by the value of transitional probability. In the case of continuous Markov processes they are usually represented by a transition rate by the transitions λ and μ from the operating and failed states respectively.
Consider now an incremental interval of time dt which is made sufficiently small so that the probability of two or more events occurring during this increment of time is negligible. This concept was first used in connection with the Poisson distribution.
The probability of being in the operating state after this time interval dt, that is the probability of being in state 0 at time (t + dt) is
[Probability of being operative at time t AND of not failing in time dt]
+ [probability of being failed at time t AND of being repaired in time dt].

Using a similar approach to that used to develop the Poisson distribution.

$$P_o(t+dt) = P_o(t)(1-\lambda dt) + P_1(t)(\mu\, dt)$$

Similarly,

$$P_1(t+dt) = P_1(t)(1-\mu dt) + P_o(t)(\lambda\, dt)$$

from equation

$$\frac{P_o(t+dt) - P_o(t)}{dt} = -\lambda P_o(t) + \mu P_1(t)$$

as dt⟶ 0

$$\left.\frac{P_o(t+dt) - P_o(t)}{dt}\right|_{dt\to 0} = \frac{dP_o(t)}{dt} = P_o^{'}(t)$$

thus,

$$P_o^{'}(t) = -\lambda P_o(t) + \mu P_1(t)$$

Similarly, from equation

$$P_1^{'}(t) = -\lambda P_o(t) + \mu P_1(t)$$

Equations may be expressed in matrix form as

$$\begin{bmatrix} P_o^{'}(t) & P_1^{'}(t) \end{bmatrix} = \begin{bmatrix} P_o(t) & P_1(t) \end{bmatrix} \begin{bmatrix} -\lambda & \lambda \\ \mu & -\mu \end{bmatrix}$$

The coefficient matrix in equation is not a stochastic transitional probability matrix because the rows of this coefficient matrix summate to zero whereas those of the stochastic transitional probability matrix summate to unity.

Equations are linear differential equations with constant coefficients. There are a number of ways in which such equations can be solved but one of the easiest and most widely used is by Laplace transforms. To illustrate this consider the Laplace transform of this

$$sP_o(s) - P_o(0) = -\lambda P_o(s) + \mu P_1(s)$$

Where $P_1(s)$ is the Laplace transform of $P_1(t)$ and $P_o(0)$ is the initial value of $P_o(t)$. Rearranging equation gives

$$P_o(s) = \frac{\mu}{s+\lambda} P_1(s) + \frac{1}{s+\lambda} P_o(0)$$

Similarly equation can be transformed into

$$P_1(s) = \frac{\lambda}{s+\mu} P_o(s) + \frac{1}{s+\mu} P_1(0)$$

Where $P_1(0)$ is the initial value of $P_1(t)$

The above equations can now be solved for $P_0(s)$ and $P_1(s)$ as linear simultaneous equations using a straightforward substitution method or using the matrix solution techniques of Appendix 3. In either case

$$P_o(s) = \frac{\mu}{\lambda+\mu}\left[\frac{P_o(0)+P_1(0)}{s}\right] + \frac{1}{\lambda+\mu}.\frac{1}{s+\lambda+\mu}[\lambda P_o(0) - \mu P_1(0)]$$

$$P_1(s) = \frac{\lambda}{\lambda+\mu}\left[\frac{P_o(0)+P_1(0)}{s}\right] + \frac{1}{\lambda+\mu}.\frac{1}{s+\lambda+\mu}[\mu P_1(0) - \lambda P_0(0)]$$

Equations above must now be transformed back into the real time domain using inverse Laplace transforms. The inverse transform of 1/s is 1 and 1/(s+a) is e^{-at}, which gives

$$P_o(t) = \frac{\mu}{\lambda+\mu}[P_o(0) + P_1(0)] + \frac{e^{-(\lambda+\mu)t}}{\lambda+\mu}[\lambda P_o(0) - \mu P_1(0)] \text{ and}$$

$$P_1(t) = \frac{\lambda}{\lambda+\mu}[P_o(0) + P_1(0)] + \frac{e^{-(\lambda+\mu)t}}{\lambda+\mu}[\mu P_1(0) - \lambda P_0(0)]$$

The term $P_o(0) + P_1(0) = 1$ for all initial conditions and therefore equations become

$$P_o(t) = \frac{\mu}{\lambda+\mu} + \frac{e^{-(\lambda+\mu)t}}{\lambda+\mu}[\lambda P_o(0) - \mu P_1(0)]$$

$$P_1(t) = \frac{\lambda}{\lambda+\mu} + \frac{e^{-(\lambda+\mu)t}}{\lambda+\mu}[\mu P_1(0) - \lambda P_0(0)]$$

In practice the most likely state in which the system starts is state 0, i.e. the system is in an operable condition at zero time. In this case

$$P_o(0) = 1 \quad and \quad P_1(0) = 0$$

and equations reduce to the frequently quoted equations for the time-dependent probabilities of a single repairable component given by

$$P_o(t) = \frac{\mu}{\lambda+\mu} + \frac{\lambda e^{-(\lambda+\mu)t}}{\lambda+\mu}$$

$$P_1(t) = \frac{\lambda}{\lambda+\mu} + \frac{\lambda e^{-(\lambda+\mu)t}}{\lambda+\mu}$$

The probabilities $P_0(t)$ and $P_1(t)$ are the probabilities of being found in the operating state and failed state respectively as a function of time given that the system started at time t=0 in the operating state.

Frequency and Duration Techniques

With the help of Markov techniques, the probability of residing in each state of the system can be found out. Similarly, the probability of being in the system up state and system down state can be evaluated from these individual state probabilities.

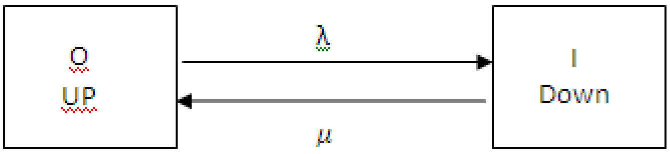
For a thorough understanding of system behavior, it is beneficial to evaluate additional reliability indices for systems.

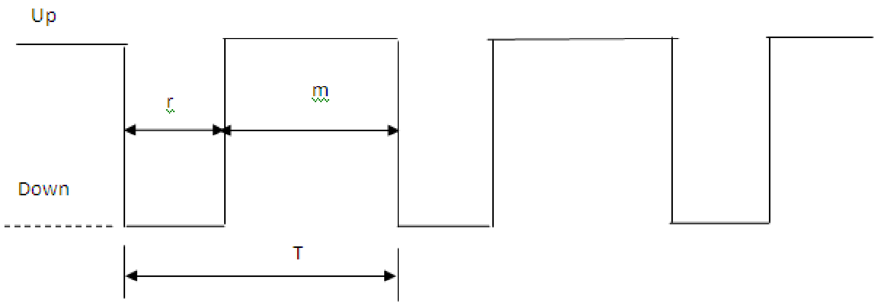
The additional indices to be evaluated are the frequency of encountering a system state and the average duration of residing in the state.

The method of deriving these additional indices can be designated as the frequency and duration technique.

Frequency & duration concepts

The basic concepts associated with the frequency and duration technique can be described in terms of single repairable component used to describe the continuous marker process.





$P_o = \frac{\mu}{\lambda + \mu}$ $\qquad$ $P_1 = \frac{\lambda}{\lambda + \mu}$

$= \frac{m}{m+r}$ $\qquad$ $= \frac{r}{m+r}$

m = MTTF = $1/\lambda$
r = MTTR = $1/\mu$
T = MTBF = $m + r = 1/f$

f = cycle frequency i.e. the frequency of encountering a system state

$$P_o = \frac{m}{m+r} = \frac{m}{T} = \frac{1}{\lambda T} = \frac{f}{\lambda}$$
$$P_1 = \frac{r}{m+r} = \frac{r}{T} = \frac{1}{\mu T} = \frac{f}{\mu}$$

Therefore frequency of encountering the 'Up' state = Po λ
= (Probability of being in the state) λ(rate of departure from the state)

Or

= $P_1\mu$ = (Probability of NOT being in the UP state) λ (rate of entry into $f = P_o\lambda = P_1\mu$ the state)

Therefore if f(s) is the frequency of encountering a state, p(s) is the probability of being in the state, P(s) is the probability of NOT being in the state, $\lambda_d(s)$ is the rate of departure from the state and $\lambda_e(S)$ is the rate of entry into the state.

$f(s) = p(s)\,\lambda_d(s) = P(s)\,\lambda_e(S)$

$$P_o = \frac{m}{m+r} = \frac{m}{T}$$

Let P(s) be the probability of residing in state S, m(s) is the mean time spent in state S & T(S) is the mean time between encounters of state S

$P(s) = \frac{m(s)}{T(s)}$

$m(s) = p(s)\,T(s)$

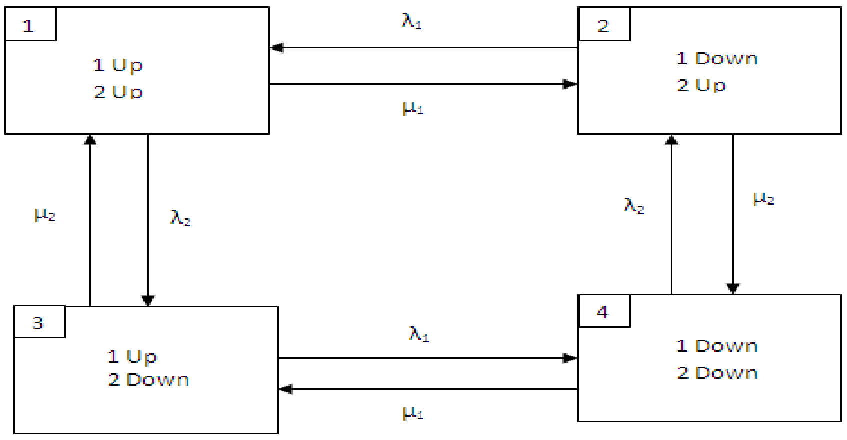
$m(s) = \frac{P(s)}{f(s)} = \frac{1}{\lambda d\,(s)}$

Mean duration of a state = reciprocal of its rate of departure

Two Component Repairable system

Consider a simple two component system, in which each component is considered to have an up state (operating) and a down state (failed) with failure and repair rates of λ_1, μ_1 and λ_2, μ_2 for components 1 and 2 respectively.

The state space diagram of this two component system is shown below



The stochastic transitional probability matrix for the system is

$$P = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 1-(\lambda_1+\lambda_2) & \lambda_1 & \lambda_2 & - \\ 2 & \mu_1 & 1-(\lambda_2+\mu_1) & - & \lambda_2 \\ 3 & \mu_2 & - & 1-(\lambda_1+\mu_2) & \lambda_1 \\ 4 & - & \mu_3 & \mu_1 & 1-(\mu_1+\mu_2) \end{array}$$

Let P_1, P_2, P_3, and P_4 be the limiting state probabilities of the states 1, 2, 3 and 4 respectively. Using the equation $\alpha P = \alpha$ where $\alpha = [P_1\ P_2\ P_3\ P_4]$ (the row vector of the state probabilities), the limiting state probabilities can be found.

$$P_1 = \frac{\mu_1\mu_2}{(\lambda_1+\mu_1)(\lambda_2+\mu_2)}$$

$$P_2 = \frac{\lambda_1\mu_2}{(\lambda_1+\mu_1)(\lambda_2+\mu_2)}$$

$$P_3 = \frac{\mu_1\lambda_2}{(\lambda_1+\mu_1)(\lambda_2+\mu_2)}$$

$$P_4 = \frac{\lambda_1\lambda_2}{(\lambda_1+\mu_1)(\lambda_2+\mu_2)}$$

If the two components are identical, then $\lambda_1 = \lambda_2 = \lambda$ and $\mu_1 = \mu_2 = \mu$, then

$$P_1 = \frac{\mu^2}{(\lambda+\mu)^2}$$

$$P_2 = \frac{2\lambda\mu}{(\lambda+\mu)^2}$$

$$P_3 = \frac{\lambda^2}{(\lambda+\mu)^2}$$

The state space diagram shown above is the same irrespective of whether the two components are in series or are parallel redundant.

In the case of a series system, state 1 is the system up state and states 2, 3 and 4 are the system down states. In the case of a parallel redundant system, states 1, 2 and 3 are the system up states and state 4 is the system down state. Therefore if P_1, P_2, P_3 and P_4 are the probabilities of being in states 1-4 respectively, then

(a) For a Series system,

$P_{up} = P_1$ $P_{down} = P_2 + P_3 + P_4$

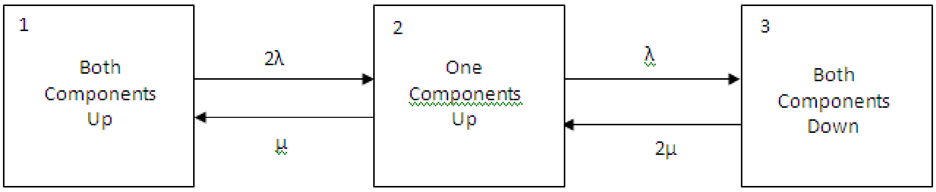
(b) For a Parallel system,

$P_{up} = P_1 + P_2 + P_3$ $P_{down} = P_4$

Two identical repairable components

In some practical situations, the state space diagram shown in figure can be simplified and reduced. For example, it may be known that when one of the components fails in a series system, the other component is no longer operating and its failure rate in these circumstances becomes zero. In this case, state 4 does not exist leaving only states 1-3 and the transition rates between these 3 states.

If both of the components are identical, states 2 and 3 are also identical and may be combined to give a reduced 3-state system as shown in below figure. The 2λ and 2μ terms indicate that two components are available for failure or repair respectively in the next increment of time and that one of the two fail or be repaired, but not both in that interval.



Then stochastic transitional probability matrix

$$P = \begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & 1-2\lambda & 2\lambda & 0 \\ 2 & M & 1-\lambda-\mu & \lambda \\ 3 & 0 & 2\mu & 1-2\mu \end{array}$$

Therefore, if the limiting state probability vector is $\{P_1\ P_2\ P_3]$, then using the equation $\alpha P = \alpha$ where $\alpha = [P_1\ P_2\ P_3]$ (the row vector of the state probabilities), the limiting state probabilities can be determined as

$$\{P_1 \quad P_2 \quad P_3] \begin{bmatrix} 1-2\lambda & 2\lambda & 0 \\ \mu & 1-\lambda-\mu & \lambda \\ 0 & 2\mu & 1-2\mu \end{bmatrix} = \{P_1 \quad P_2 \quad P_3]$$

$$P_1(1-2\lambda) + P_2\mu = P_1$$
$$P_1 2\lambda + P_2(1-\lambda-\mu) + P_3 2\mu = P_2$$
$$P_2\lambda + P_3(1-2\mu) = P_3$$

Rearranging gives

$$-2\lambda P_1 + \mu P_2 = 0$$
$$2\lambda P_1 - (\lambda + \mu)P_2 + 2\mu P_3 = 0$$
$$\lambda P_2 - 2\mu P_3 = 0$$
$$P_1 + P_2 + P_3 = 1$$

The limiting state probabilities are

$$P_1 = \frac{\mu^2}{(\lambda+\mu)^2} \qquad P_2 = \frac{2\lambda\mu}{(\lambda+\mu)^2} \qquad P_3 = \frac{\lambda^2}{(\lambda+\mu)^2}$$

(a) Series connected components

In the case of two identical components connected in series, the up state of the system is state 1 and the down state is states 2 and 3, therefore

Availability $A = P_1$

$$= \frac{\mu^2}{(\lambda+\mu)^2}$$

Unavailability, $U = P_2 + P_3$

$$= \frac{2\lambda\mu + \lambda^2}{(\lambda+\mu)^2}$$

(b) Parallel connected components

In the case of two identical components connected in parallel, state 2 also becomes an up state giving

Availability, $A = P_1 = P_2$

$$= \frac{\mu^2 + 2\lambda_\mu}{(\lambda+\mu)^2}$$

Unavailability, $U = P_3$

$$= \frac{\lambda^2}{(\lambda+\mu)^2}$$

It is interesting to note that, in this system, each component is independent, and the expressions for P_1, P_2 and P_3 together with the values of availability and unavailability could have been obtained directly from the results of the single component system, equations using the binomial expansion. This applies to any number of independent components and the reader should therefore be able to derive the equivalent equations for limiting state probabilities for a three component system from the binomial expansion and verify these using the technique described in this section.

Frequency of encountering individual states

The second step in the evaluation of the frequency and duration indices of a system is to evaluate the frequency of encountering the individual states. The rates of departure and entry to each state can be identified from both the state space diagram and also from the stochastic transitional probability matrix P.

Rates of departure and entry

State number	Component 1	Component 2	Rate of Departure	Entry
1	Up	Up	$\lambda_1 + \lambda_2$	$\mu_1 + \mu_2$
2	Down	Up	$\mu_1 + \lambda_2$	$\lambda_1 + \mu_2$
3	Up	Down	$\lambda_1 + \mu_2$	$\lambda_2 + \mu_1$
4	Down	Down	$\mu_1 + \mu_2$	$\lambda_1 + \lambda_2$

(a) Frequency of encountering state 1

From equation, if f_1 = frequency of encountering state 1

= P_1 X (rate of departure from state 1)

$$= \frac{\mu_1\mu_2(\lambda_1+\lambda_2)}{(\lambda_1+\mu_1)(\lambda_2+\mu_2)}$$

Also from equation

$f_1 = \overline{P_1}$ x (rate of entry to state 1)

The application of this form of the concept must be treated with care since it applies only to the communicating states. Therefore, in the case of state 1, the only communicating states are 2 and 3, the rates of entry to state 1 from these being μ_1 and μ_2 respectively.

Thus $f_1 = P_2\,\mu_1 + P_3\,\mu_2$

$$= \frac{\mu_1\mu_2(\lambda_1+\lambda_2)}{(\lambda_1+\mu_1)(\lambda_2+\mu_2)}$$

(b) Frequency of encountering state 4

f_4 = frequency of encountering state 4

= P_4 x (rate of departure from state 4)

$$= \frac{\lambda_1\lambda_2(\mu_1+\mu_2)}{(\lambda_1+\mu_1)(\lambda_2+\mu_2)}$$

or $f_4 = P_2\,\lambda_2 + P_3\lambda_1$

$$= \frac{\lambda_1\lambda_2(\mu_1+\mu_2)}{(\lambda_1+\mu_1)(\lambda_2+\mu_2)}$$

Similar derivations may be made for f_2 and f_3, the frequencies of encountering states 2 and 3 respectively. The complete list of state probabilities and frequencies of encounter are shown in below table.

State number	Probability	Frequency of encounter
1	$\mu_1\mu_2/D$	$\mu_1\mu_2(\lambda_1 + \lambda_2)/D$
2	$\lambda_1\mu_2/D$	$\lambda_1\mu_2(\mu_1 + \lambda_2)/D$
3	$\mu_1\lambda_2/D$	$\mu_1\lambda_2(\lambda_1 + \mu_2)/D$
4	$\lambda_1\lambda_2/D$	$\lambda_1\lambda_2(\mu_1 + \mu_2)/D$

Where $D = (\lambda_1 + \mu_1)(\lambda_2 + \mu_2)$

In the case when both components are identical components, that is, $\lambda_1 = \lambda_2 = \lambda$ and $\mu_1 = \mu_2 = \mu$

$$f_1 = \frac{2\lambda\mu^2}{(\lambda+\mu)^2}$$

$$f_2 = f_3 = \frac{\lambda\mu}{\lambda+\mu}$$

$$f_4 = \frac{2\lambda^2\mu}{(\lambda+\mu)^2}$$

Mean duration of individual states

Using the concept, the mean duration of each of the system states, i.e., the mean time of residing in each of the states, can be evaluated directly and simply from the rates of departure. Therefore if m_1, m_2, m_3 and m_4 are defined as the mean duration of states 1-4 respectively, then

$$m_1 = \frac{1}{\lambda_1 + \lambda_2}$$

$$m_2 = \frac{1}{\lambda_2 + \mu_1}$$

$$m_3 = \frac{1}{\lambda_1 + \mu_2}$$

$$m_4 = \frac{1}{\mu_1 + \mu_2}$$

Which, for identical components

$$m_1 = \frac{1}{2\lambda} \qquad m_2 = m_3 = \frac{1}{\lambda+\mu} \qquad m_4 = \frac{1}{2\mu}$$

The value of m_1 has already been deducted as the MTTF or mean up time of a series system containing two identical components. Similarly the value m_4 is the mean down time or MTTR of a parallel system.

Cycle time between individual states

The cycle time T is the reciprocal of the frequency of encounter f. A cycle time for each individual state can therefore be deducted from the frequency of encountering this state. This value of cycle time represents the mean time between entering a given state to next entering the same state. In the case of the two component system and considering the case of identical components

$$T_1 = \frac{(\lambda + \mu)^2}{2\lambda\mu^2}$$

$$T_2 = T_3 = \frac{\lambda + \mu}{\lambda\mu} = \frac{1}{\lambda} = \frac{1}{\mu}$$

$$T_4 = \frac{(\lambda + \mu)^2}{2\lambda^2\mu}$$

Consider the two cases of parallel redundant system and series systems.

(a) Parallel redundant systems

The MTTF of a parallel redundant system is defined as the mean time between encounters of the state in which both components are down.
The MTTF is given in equation as

MTTF = $\frac{2\lambda+\mu}{2\lambda^2}$

And the MTTR is given by m_4 in equation
Since the MTBF of a parallel redundant system is given by the sum of the MTTF and MTTR, then
MTBF = $\frac{2\lambda+\mu}{2\lambda^2} + \frac{1}{2\mu}$
= $\frac{\lambda^2+2\lambda\mu+\mu^2}{2\lambda^2\mu}$
= $\frac{(\lambda+\mu)^2}{2\lambda^2\mu}$
Which is identical for the cycle time T4 using the frequency and duration method. The latter technique does not require the concept of absorbing states and is generally a simpler method to use. The two results are expected to be identical because the MTBF of a parallel redundant system is equivalent to the system cycle time of encountering state 4, i.e., the state in which both components are failed.

(b) Series system

In the case of series system the MTTF is $1/2\lambda$ and the MTTR is $(\lambda+2\mu)/2\mu^2$. The value of MTTR was not derived but can be verified by truncating state 1 and solving for M

Therefore

MTBF = $\frac{1}{2\lambda} + \frac{\lambda+2\mu}{2\mu^2}$
= $\frac{(\lambda+\mu)^2}{2\lambda\mu^2}$

which is identical to the cycle time T_1 given by equation. Again, these two results are expected to be identical since the up state of the series system is state 1 when both components are up, and the MTBF of the system is given by the mean time of encountering this state.

Frequency of encountering cumulated states

In most system reliability evaluation problems, the frequency, duration and cycle time of individual states only provide a partial answer to the problem. As discussed there may be several states of the system which have a similar impact on the system behavior. States leading to the same system outcome can be combined or cumulated to give, for example, the system up state, system down state and system derated states (if any).

The probability of residing in one of these cumulated states can be evaluated by simply summating the mutually exclusive probabilities of each appropriate state. A Similar technique can be used to evaluate the frequency of encountering cumulated states 3 and 4. Define the cumulative frequency of combining states 3 and 4 as f_{34}.

The transition frequencies to be included in f_{34} must include the frequencies of all transitions that leave and enter the combined state (3 and 4) but most ignore all transition frequencies that occur between states 3 and 4 since these do not represent transitions out of the combined state (3 and 4). Therefore

$$
\begin{aligned}
f_{34} &= f_3 + f_4 - (\text{frequency of encounters between 3 and 4}) \\
&= f_3 + f_4 - (P_3\lambda_1 + P_4\,\mu_1) \\
&= P_3(\lambda_1 + \mu_2) + P_4(\mu_1 + \mu_2) - P_3\lambda_1 - P_4\,\mu_1 \\
&= P_3\,\mu_2 + P_4\mu_2 \\
&= (P_3 + P_4)\,\mu_2
\end{aligned}
$$

The above equation illustrates an important underlying principle. The frequency of encountering the cumulated state (3 and 4) can be obtained by considering the expected number of transitions across the boundary wall surrounding the cumulated state. This has two components $P_3\,\mu_2$ and $P_4\,\mu_2$ arising from states 3 and4, respectively. The frequency is therefore

$$
\begin{aligned}
f_{34} &= P_3\,\mu_2 + P_4\,\mu_2 \\
&= (P_3 + P_4)\,\mu_2
\end{aligned}
$$

Substituting for P_3 and P_4 in equation gives

$$f_{34} = \frac{\mu_2\lambda_2}{\mu_2+\lambda_2}$$

Similarly

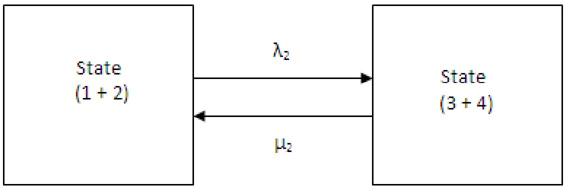
f_{12} = frequency of encountering the combined state (1 and 2)

$= f_1 + f_2 - P_1\lambda_1 - P_2\mu_1$

$= (P_1 + P_2)\lambda_2$

$= \frac{\mu_2\lambda_2}{\mu_2+\lambda_2}$

It can be seen that equations are identical. This is to be expected since the system has been effectively reduced to two cumulated states (1 and 2) and (2 and 4), in which case the frequency of encountering each must be the same. The effective state space diagram of this system is shown in below figure



UNIT 5

Approximate System Reliability Evaluation

The Markov Technique and the frequency and duration approach form sound and precise modeling and evaluation methods for reliability applications.

- They become less amenable, for hand calculations and even for digital computer solutions as the system becomes larger and more complex.
- In such cases, alternative methods are available which are based on the Markov approach and which use a set of appropriate but approximate equations.
- The essence of these approximate techniques is to derive a set of equations suitable for a series system in which all components must operate for system success and for a parallel system in which only one component need work for system success.

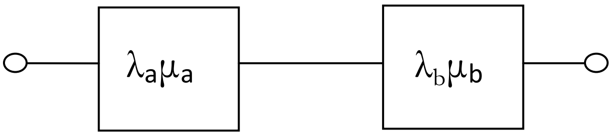
Series Systems

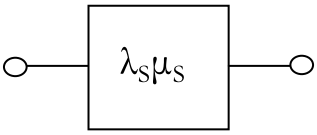
Consider two components connected in series. The state space diagram for this system is shown below assuming that all states can exist.

The probability of the system being in the up state i.e. both components operating is given by

$$P_{up} = \frac{\mu_1\mu_2}{(\lambda_1+\mu_1)(\lambda_2+\mu_2)} \longrightarrow \quad (1)$$

It is necessary to find the failure and repair rates λ_s and μ_s of a single component that is equivalent to the two components in series. This is shown in figure





The probability of single component being in up state is

$$P_{up} = \frac{\mu_s}{(\lambda_s+\mu_s)} \qquad \longrightarrow \qquad (2)$$

For the single component to be equivalent to the two series components

$$\frac{\mu_1\mu_2}{(\lambda_1+\mu_1)(\lambda_2+\mu_2)} = \frac{\mu_s}{\lambda_s+\mu_s} \qquad \longrightarrow \qquad (3)$$

Since the transition rate from the system up state for the single equivalent component is λ_s, and for two component series system is $\lambda_a + \lambda_b$, then

$$\lambda_s = \lambda_a + \lambda_b \qquad \longrightarrow \qquad (4)$$

Substituting (4) into (3) and replacing the repair rates μ_s by the reciprocal of the average repair times, r_s gives

$$r_s = \frac{1}{\mu_s} = \frac{\lambda_a r_a + \lambda_b r_b + \lambda_a \lambda_b r_a r_b}{\lambda_s} \qquad \longrightarrow \qquad (5)$$

In many systems the product $\lambda_i r_i$ is very small and therefore $\lambda_a\lambda_b r_a r_b$ is very much less than $\lambda_a r_b$ and $\lambda_b r_b$

In such cases equation (5) reduces to

$$r_s = \frac{\lambda_a r_a + \lambda_b r_b}{\lambda_s} \qquad \longrightarrow (6)$$

Equation (6) is an approximation for a two component series system in which all four states exist, it is an exact expression for the situation in which state (4) does not exist i.e, when one component has failed the second component cannot fail.

This occurs in practice when, after failure of the first component, the failure rates for the remaining operative but not-working components either decrease to zero or become negligible.

Using the logic expressed in equations (4) and (6), the failure rate and average outage duration of a general n-component series system is defined as

$$\lambda_s = \sum_{i=1}^{n} \lambda_i$$

$$r_s = \frac{\sum_{i=1}^{n} \lambda_i \, r_i}{\lambda_5}$$

The probability of the system being in the down state, i.e the unavailability Vs can be related to r_s and the frequency of encountering the down state f_s using the concepts of frequency and duration is $V_s = f_s r_s$

MTTF = $1/\lambda$ and MTBF = $1/f$ are conceptually different although for many practical systems they are numerically almost identical

$$V_s \approx \lambda_s r_s = \sum_{i=1}^{n} \lambda_i r_i$$

If the units of time for λ_s and r_s are the same, the value of V_s is strictly a probability. If the units are different, e.g λ_s is expressed in failures per year and r_s is expressed in hours and the value of V_s has dimensional units associated with it, e.g hours per year.

Therefore, the set of equations frequency used for a series system is

$$\lambda_s = \sum_{i=1}^{n} \lambda_i$$

$$r_s = \frac{\sum_{i=1}^{n} \lambda_i \, r_i}{\lambda_5}$$

$$= \frac{V_s}{\lambda_s}$$

$$V_s = \lambda_s r_s$$

$$= \sum_{i=1}^{n} \lambda_i r_i$$

Repair time is usually very small compared with operating time and hence

MTTF ≈ MTTBF

The failure rates of three components are 0.05 f/yr, 0.01 f/yr and 0.02 f/yr respectively and their average repair times are 20hr, 15hr and 25hr respectively. Evaluate the system failure rate, average repair, time and unavailability, if all three components must operate for system success.

$\lambda_S = 0.05 + 0.01 + 0.02 = 0.08$ f/yr
$V_s = 0.05 \text{ X } 20 + 0.01 \text{ X } 15 + 0.02 \text{ X } 25 = 1.65$ hr/yr
$R_s = 1.65/0.08 = 20.6$ hr

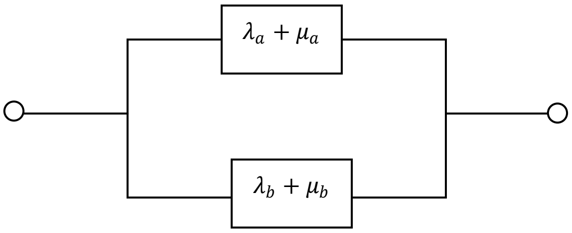
Parallel Systems

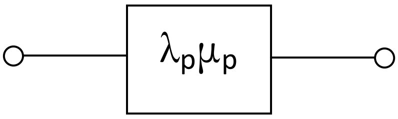
Two Component System

The probability of the system being in the down state is given by

$$p_{down} = \frac{\lambda_a \lambda_b}{(\lambda_a + \mu_a)(\lambda_b + \mu_b)}$$

In this case the failure rate λ_p and repair rate μ_p of a single component that is equivalent to the two components in parallel is required.





The probability of single component is down state is

$$p_{down} = \frac{\lambda_p}{\lambda_p + \mu_p}$$

$$\frac{\lambda_p}{\lambda_p + \mu_p} = \frac{\lambda_a \lambda_b}{(\lambda_a + \mu_a)(\lambda_b + \mu_b)}$$

The rate of transition from the down state of the two component system is $\mu_a + \mu_b$ and this must be equivalent to μ_p

$$\mu_P = \mu_a + \mu_b$$

$$\frac{1}{r_p} = \frac{1}{r_a} + \frac{1}{r_b}$$

$$r_p = \frac{r_a r_b}{r_a + r_b}$$

Problems:

A system consists of two components, one of which must operate for system success. If the failure rates are 0.05 f/yr and 0.02 f/yr respectively and the average repair times are 20hr and 25hr respectively, evaluate the system failure rate, average repair time and unavailability.

Solution:

$$\lambda_p = 0.05 \text{ X } \frac{0.02\ (20+25)}{8760}$$

$$= 5.14 \text{ X } 10^{-6} \text{ f/yr}$$

$$r_p = \frac{20+25}{20+25} = 11.1\ hr$$

$$U_p = \lambda_p r_p = 5.71 \text{ X } 10^{-5} \text{ hr/yr}$$

This value of r_p represents the average period of time during which both components are concurrently out of service, i.e it represents the period during which the two failures overlap.

For this reason, r_p is generally known as the overlapping repair or outage time of components a and b. The failure event caused by the failure of components a and b is generally known as an overlapping failure event.

$$\frac{\lambda_p}{\lambda_p + \mu_p} = \frac{\lambda_1\lambda_2}{(\lambda_1 + \mu_1)(\lambda_2 + \mu_2)}$$

$\lambda_P(\lambda_1\lambda_2 + \lambda_1\mu_2 + \mu_1\lambda_2 + \mu_1\mu_2) = \lambda_1\lambda_2\lambda_p + \lambda_1\lambda_2\mu_p$

$\lambda_P(\lambda_1\lambda_2 + \lambda_1\mu_2 + \mu_1\lambda_2 + \mu_1\mu_2 - \lambda_1\lambda_2) = \lambda_1\lambda_2\mu_p$

$$\lambda_p = \frac{\lambda_1\lambda_2\mu_p}{\lambda_1 - \lambda_2 + \mu_1\lambda_2 + (\mu_1 - \mu_2)}$$

$$= \frac{\lambda_1\lambda_2(r_p + r_2}{(\lambda_1 - \mu_2 + \mu_1\lambda_2 + \mu_1\mu_2)r_1 r_2}$$

$$= \frac{\lambda_1\lambda_2(r_1 + r_2)}{\lambda_1 r_1 + \lambda_2 r_2 + 1}$$

$$\lambda_p = \frac{\lambda_a\lambda_b(r_a + r_b)}{1 + \lambda_a r_a + \lambda_b r_b}$$

If $\lambda_a r_a$ and $\lambda_b r_b$ are much less than unity, then

$$\lambda_P \approx \lambda_a\lambda_b\ (r_a + r_b)$$

$$U_p = \lambda_p r_p = \lambda_a\lambda_b r_a r_b$$

Probability of $U_p = f_p/\mu_p$

$$= \lambda_p r_p = \lambda_a \lambda_b (r_a + r_b)\, X \frac{r_a r_b}{r_a + r_b} = \lambda_a \lambda_b r_a r_b$$

Series System	**Parallel System (Two Component)**	**Parallel System (Three Component)**
$\lambda_s = \sum_{i=1}^{n} \lambda_i$ $r_s = \frac{\sum_{i=1}^{n} \lambda_i r_i}{\lambda_5}$ $= \frac{V_s}{\lambda_s}$ $V_s = \lambda_s r_s$ $= \sum_{i=1}^{n} \lambda_i r_i$	$r_p = \frac{r_a r_b}{r_a + r_b}$ $\lambda_p = \lambda_a \lambda_b (r_a + r_b)$ Probability of U_p = $U_p = \lambda_p r_p = \lambda_a \lambda_b r_a r_b$	$r_p = \frac{r_a r_b r_c}{r_a r_b + r_b r_c + r_c r_a}$ $\lambda_p = \lambda_a \lambda_b \lambda_c (r_a r_b + r_b r_c + r_c r_a)$ $U_p = \lambda_p r_p = \lambda_a \lambda_b \lambda_c r_a r_b r_c$

System with more than two components

Unlike the case of series systems, it is not possible to extend easily the equations for a 2-component parallel system to a general n-component system.

It is possible in certain systems to combine two components at a time using equations. This method must be treated with the utmost care because it becomes invalid if the concepts of a single failure rate per component, or a single environmental state, are extended to more complex situations. It is better to use an appropriate set of equations for the number of components that require combining.

It is evident tat such equations can be deduced from first principles using the concepts. It is simpler to deduce the equations from the logic of equations. In order to understand this logic, rewrite equation in the form

This equation may be expressed in words as:

'failure of the system occurs if (component 1 fails followed by failure of component 2 during the repair time of component 1) or (component 2 fails followed by failure of component 1 during the repair time of component 2)'.

The product terms contained by the parentheses in equation represent the probability that one component fails during the outage time of the other; this condition being the only way in which a parallel system can fail. The 'time' parameter within the parentheses is generally known as he 'exposure time' during which the associated component must fail for the system to fail

Using this logic the expression for the failure rate of a system containing three or more components can be deduced. Consider for example, the case of a three component (A, B and C) parallel system. In this case the failure expression is:

'failure of the system occurs if
(A fails followed by failure of B during repair of A followed by failure of C during the overlapping repair of A and B) OR
(A fails followed by failure of C during repair of A followed by failure of B during the overlapping repair of A and C) OR
(Plus 4 more similar statements for the failure sequences BAC, BCA, CAB, CBA)'
Which, expressed in mathematical form and using equation, gives

$$\lambda_p = \lambda_A(\lambda_B r_A)\left(\lambda_C \frac{r_A r_B}{r_A + r_B}\right) + \lambda_A(\lambda_C r_A)\left(\lambda_B \frac{r_A r_C}{r_A + r_C}\right)$$

$$+\lambda_B(\lambda_A r_B)\left(\lambda_C \frac{r_A r_B}{r_A + r_B}\right) + \lambda_B(\lambda_C r_B)\left(\lambda_A \frac{B r_C}{r_B + r_C}\right)$$

$$+\lambda_C(\lambda_C r_C)\left(\lambda_B \frac{r_A r_C}{r_A + r_C}\right) + \lambda_C(\lambda_B r_C)\left(\lambda_A \frac{r_B r_C}{r_B + r_C}\right)$$

$= \lambda_A\lambda_B\lambda_C(r_Ar_B+r_Br_C+r_Cr_A)$

Also, from the logic of equation

$$\mu_P = \mu_A + \mu_B + \mu_C$$

$$\frac{1}{r_p} = \frac{1}{r_A} + \frac{1}{r_B} + \frac{r}{r_C}$$

$$r_p = \frac{r_A r_B r_c}{r_A r_B + r_B r_C + r_C r_A}$$

and $U_p = \lambda_p r_p = \lambda_A \lambda_B \lambda_C r_A r_b r_C$

Network reduction techniques

Most systems do not consist of only series chain or parallel configurations but more often a combination of both. The general principles and concepts of such networks were discussed and it is therefore not intended to repeat these details at this point. Briefly, however, one method for solving these networks is sequentially to reduce the network using appropriate equations for series and parallel combinations until the network is reduced to a single equivalent component. This method, known as network reduction, was described and the reliability parameters of the equivalent component are the parameters of the complete system.

Consider the following numerical example to illustrate the application of the series and parallel equations derived to the network reduction technique.

Reconsider and evaluate the system failure rate, average repair time and unavailability if all components are identical and have a failure rate of 0.05 f/yr and an average repair time of 20 hours.

The first reduction requires combining components 3 and 4 in parallel to give equivalent component 6. Using equation gives

$$\lambda_6 = 0.05 \text{ X } \frac{0.05\ (20+20)}{8760}$$

$$= 1.14 \text{ X } 10^{-5} \text{ f/yr}$$

$$r_6 = \frac{20\ X\ 20}{20+20} = 10\ hr$$

The second reduction requires combining components 1, 2 and 6 in series to give equivalent component 7.

$$\lambda_7 = 0.05 + 0.05 + 1.14 \text{ X } 10^{-5} = 0.10 \text{ f/yr}$$

$$\lambda_7 = \frac{0.05\ X\ 20 + 0.05\ X\ 20 + 1.14\ X\ 10^{-5}\ X\ 10)}{0.10} = 20hr$$

The final reduction requires combining components 5 and 7 in parallel to give equivalent component 8 which then represents the system indices. Using equations gives

$$\lambda_8 = 0.05 \text{ X } \frac{0.10\ (20+20)}{8760}$$

$$= 2.28 \text{ X } 10^{-5} \text{ f/yr}$$

$$r_8 = \frac{20\ X\ 20}{20+20} = 10\ hr$$

$$U_8 = 2.28 \times 10^{-4} \text{ hr/yr}$$

This example shows that a series / parallel system can be evaluated by sequential application of the series and parallel equations. This method however cannot be used directly if the system is more complex, i.e a non-series/parallel configuration such as the bridge network shown. Some authors have suggested that such a network can be transformed into one containing only series/parallel branches using a method known as the star-delta transformation. This method can become quite tedious and the minimal cut set technique is usually preferable. The advantages of the minimal cut set approach are that it eliminates the need for complicated transformation, and it directly indicates the predominant failure modes of the system. The importance of retaining a physical appreciation of the system and its failure modes is a fundamental requirement in overall system reliability evaluation.

Minimal cut set/failure modes approach

The minimal cut set method was described in detail and will not be discussed at length here. However, it should be recalled that it enables a reliability network, expressed in terms of minimal cut sets, to be deduced from the system operational logic and/or system network diagram. This reliability network consists of a number of minimal cut sets connected in series and each cut set consists of a number of components connected in parallel. It follows that the series/parallel equations derived can be applied directly to a minimal cut set diagram. The procedure is to apply the equations for parallel systems to each cut set in order to evaluate the equivalent indices for each cut set and then to combine these equivalent indices using the equations for series systems to give the overall system reliability indices.

In order to illustrate the application of the equations to the minimal cut set method

A visual inspection identifies two second-order and one third-order cut sets.

After evaluating the reliability indices for each cut set, can be used to evaluate the system indices. This is most conveniently accomplished by summating the values of λ to give λ_s, by summating the values of U to give U_s and then evaluating r_s by dividing the value of U_s by λ_s. These system indices are shown and can be compared with those obtained previously using network reduction.

The following comments can be made in the light of these results and the analysis used to achieve them:

(a) The system indices are generally dominated by the low order cut sets which, in the case of the above example, are the two second order cuts. Sufficient precision is therefore generally achieved by ignoring cut sets that are more than one or two orders greater than the lowest order cut sets that exist. It must be stressed that this assumption may not be as valid if the components forming the lower order cuts are very reliable and the components forming the higher order cuts are very unreliable.

(b) The minimal cut sets of the system define directly the failure modes of the system. The system will fail in the above example if components 1 and 5 are failed or if 2 and 5 are failed or if 3 and 4 and 5 are failed. In addition the method quantities the impact of each failure event on the system in terms of λ, r and U. Therefore, it is possible to determine from this analysis not only the system reliability indices but also the contributions made to the system indices by the various failure modes. This information is very important in any properly structured reliability assessment since it identifies critical areas of system weakness and suggests where reinforcement and investment should be made. None of this information is readily obtained from the network reduction method.

(c) From an assessment of this type, it may be decided that reinforcement and investment should be made